Taylor model methods vs. interval methods for ODEs: Similarities and distinctions

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Outline

Verified Integration of ODEs

Interval IVP:

$$u' = f(t,u), \quad u(t_0) \in \mathbf{u}_0, \quad t \in \mathbf{t} = [t_0, t_{\text{end}}]$$

$f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ sufficiently smooth, $\quad \mathbf{u}_0 \in \mathbb{IR}^m, \quad t_{\text{end}} > t_0$.

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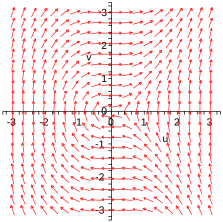

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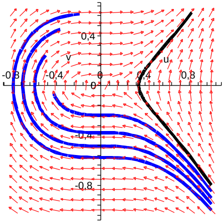

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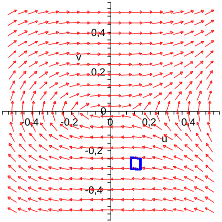

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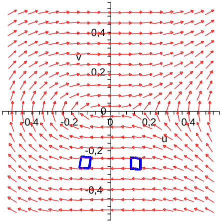

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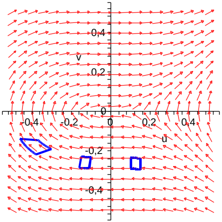

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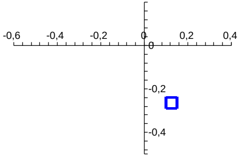

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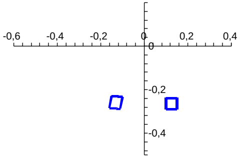

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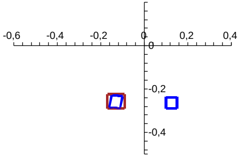

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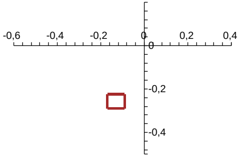

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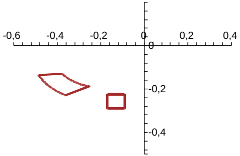

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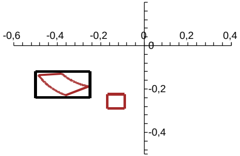

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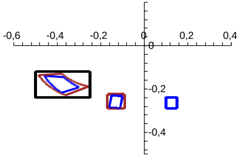


Interval Methods for ODEs

Introduction

Set of autonomous IVPs:

$$u' = f(u), \quad u(t_0) = u_0 \in \mathbf{u}_0, \quad t \in \mathbf{t} = [t_0, t_{\text{end}}],$$

where $D \subset \mathbb{R}^m$, $f \in C^n(D)$, $f : D \to \mathbb{R}^m$, $\mathbf{u}_0 \in \mathbb{IR}^m$.

Moore's enclosure method:

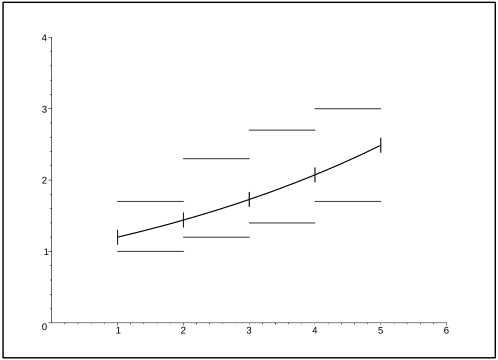
- Automatic computation of Taylor coefficients.
- Interval iteration: For $j := 1, 2, \ldots$:

A priori enclosure: $\mathbf{v}_j \supseteq u(t)$ for all $t \in [t_{j-1}, t_j]$ ("Alg. I").

Truncation error: $$\mathbf{z}_j := \frac{h_j^{(n+1)}}{(n+1)!} f^{(n)}(\mathbf{v}_j).$$

$$u(t_j) \in \mathbf{u}_j := \mathbf{u}_{j-1} + \sum_{k=1}^{n} \frac{h_j^k}{k!} f^{(k-1)}(\mathbf{u}_{j-1}) + \mathbf{z}_j \quad \text{("Algorithm II")}.$$

Piecewise constant a priori enclosure



A priori Enclosures

- Picard iteration: find h_j, $\mathbf{v}_j$ such that

$$\mathbf{u}_{j-1} + [0, h_j] f(\mathbf{v}_j) \subseteq \mathbf{v}_j$$

- Step size restrictions: Explicit Euler steps
- More advanced schemes: Lohner 1988, Corliss & Rihm 1996, Nedialkov & Jackson 2001, Makino 1998.

Modifications of Algorithm II

- Reduction of wrapping effect: Moore, Eijgenraam, Lohner, Rihm, Kuehn, Nedialkov & Jackson, ...
- Nedialkov & Jackson: Hermite-Obreshkov-Method
- Rihm: Implicit methods
- Petras & Hartmann, Bouissou: Runge-Kutta-Methods
- Berz & Makino: Taylor models

  Taylor expansion with respect to initial values and parameters

Direct method

Direct method (Moore 1965): Apply mean value form to $f^{(k)}$:

$$\widehat{u}_j \in \mathbf{u}_j, \quad f^{[0]}(u) = u, \quad f^{[k]}(u) = \frac{1}{k}\left(\frac{\partial f^{[k-1]}}{\partial u} f\right)(u) \;\text{ for }\; k \geq 1.$$

Let

$$\mathbf{S}_j = I + \sum_{k=1}^{n} h_0^k J\big(f^{[k]}(\mathbf{u}_{j-1})\big), \quad \mathbf{z}_j = h_0^{n+1} f^{[n]}(\mathbf{v}_j),$$

(I: identity matrix, $J\big(f^{[k]}\big)$: Jacobian of $f^{[i]}$), then

$$u(t_j; u_0) \in \mathbf{u}_j = \widehat{u}_{j-1} + \sum_{k=1}^{n-1} h_j^k f^{[k]}(\widehat{u}_{j-1}) + \mathbf{z}_j + \mathbf{S}_j(\mathbf{u}_{j-1} - \widehat{u}_{j-1}).$$

Interval Methods for ODEs

Wrapping effect: $\mathbf{S}_j(\mathbf{u}_{j-1} - \widehat{u}_{j-1})$ may overestimate

$$\mathcal{S} = \{S_j(u_{j-1} - \widehat{u}_{j-1}) \mid S_j \in \mathbf{S}_{j-1},\ u_{j-1} \in \mathbf{u}_{j-1}\}$$

Reduction: propagate $\mathcal{S}$ as a parallelepiped.
Let $\widehat{u}_0 := \mathrm{m}(\mathbf{u}_0)$, $\mathbf{r}_0 = \mathbf{u}_0 - \widehat{u}_0$, $B_0 = I$.
For some nonsingular matrices B_j, do:

$$\left.\begin{aligned} \widehat{u}_j &= \widehat{u}_{j-1} + \sum_{k=1}^{n-1} h_{j-1}^k f^{[k]}(\widehat{u}_{j-1}) + \mathrm{m}(\mathbf{z}_j), \\ \mathbf{u}_j &= \widehat{u}_{j-1} + \sum_{k=1}^{n-1} h_{j-1}^k f^{[k]}(\widehat{u}_{j-1}) + \mathbf{z}_j + (\mathbf{S}_{j-1} B_{j-1})\mathbf{r}_{j-1}, \end{aligned}\right\}$$

$\widehat{u}_j$: approximate point solution for the central IVP
$\mathbf{z}_j$: local error; $\mathbf{r}_j$: global error

Global error

Global error propagation:

$$\mathbf{r}_j = \left(B_j^{-1}(\mathbf{S}_{j-1}B_{j-1})\right)\mathbf{r}_{j-1} + B_j^{-1}(\mathbf{z}_j - \mathrm{m}(\mathbf{z}_j))$$

- Moore's direct method: $B_j = I$
- Pep method (Eijgenraam, Lohner): $B_j = \mathrm{m}(\mathbf{S}_{j-1}B_{j-1})$
- QR method (Lohner): B_j is chosen as the orthogonal matrix Q in the QR factorization of $\mathrm{m}(\mathbf{S}_{j-1}B_{j-1})$
- Blunting method (Berz, Makino): modify B_j in the pep method such that condition numbers remain small

Taylor Model Methods for ODEs

Quadratic Model Problem

$$\begin{aligned} u' &= v, & u(0) &\in [0.95, 1.05], \\ v' &= u^2, & v(0) &\in [-1.05, -0.95]. \end{aligned}$$

Taylor model method: initial set described by parameters a and b:

$$\begin{aligned} u_0(a,b) &:= 1 + a, & a &\in \mathbf{a} := [-0.05, 0.05], \\ v_0(a,b) &:= -1 + b, & b &\in \mathbf{b} := [-0.05, 0.05]. \end{aligned}$$

Naive Taylor Model Method

Picard iteration ($n = 3,\ h = 0.1$):

$$u^{(0)}(\tau, a, b) = 1 + a, \quad v^{(0)}(\tau, a, b) = -1 + b,$$

$$u^{(1)}(\tau, a, b) = u_0(a, b) + \int_0^\tau v^{(0)}(s, a, b)\, ds$$
$$v^{(1)}(\tau, a, b) = v_0(a, b) + \int_0^\tau \left(u^{(0)}(s, a, b)\right)^2\, ds$$

$$u^{(3)}(\tau, a, b) = 1 + a - \tau + b\tau + \tfrac{1}{2}\tau^2 + a\tau^2 - \tfrac{1}{3}\tau^3,$$
$$v^{(3)}(\tau, a, b) = -1 + b + \tau + 2a\tau - \tau^2 + a^2\tau - a\tau^2 + b\tau^2 + \tfrac{2}{3}\tau^3.$$

Naive Taylor Model Method: Remainder Bounds

Remainder bounds by fixed point iteration (Makino, 1998):

Find $\mathbf{i}_0$ and $\mathbf{j}_0$ s.t.

$$u_0 + \int_0^\tau \left(v^{(3)}(s,a,b) + \mathbf{j}_0\right)\, ds \quad \subseteq \quad u^{(3)}(\tau,a,b) + \mathbf{i}_0,$$

$$v_0 + \int_0^\tau \left(u^{(3)}(s,a,b) + \mathbf{i}_0\right)^2\, ds \quad \subseteq \quad v^{(3)}(\tau,a,b) + \mathbf{j}_0$$

for all $a \in \mathbf{a}$, $b \in \mathbf{b}$, $\tau \in [0, 0.1]$.

Naive Taylor Model Method: Enclosure of the Flow

Flow for $\tau \in [0, 0.1]$:

$$\begin{aligned} \widetilde{\mathcal{U}}_1(\tau, a, b) &:= u^{(3)}(\tau, a, b) + \mathbf{i}_0, \\ \widetilde{\mathcal{V}}_1(\tau, a, b) &:= v^{(3)}(\tau, a, b) + \mathbf{j}_0. \end{aligned}$$

Flow at $t_1 = 0.1$:

$$\begin{aligned} \mathcal{U}_1(a, b) &:= \widetilde{\mathcal{U}}_1(0.1, a, b) = 0.905 + 1.01a + 0.1b + \mathbf{i}_0, \\ \mathcal{V}_1(a, b) &:= \widetilde{\mathcal{V}}_1(0.1, a, b) = -0.909 + 0.19a + 1.01b + 0.1a^2 + \mathbf{j}_0 \end{aligned}$$

(nonlinear boundary).

Naive Taylor Model Method: Second Integration Step

Find $\mathbf{i}_1$ and $\mathbf{j}_1$ s.t.

$$\begin{aligned} \mathcal{U}_1(a,b) + \int_0^\tau \left(v^{(3)}(s,a,b) + \mathbf{j}_1\right) ds &\subseteq u^{(3)}(\tau,a,b) + \mathbf{i}_1, \\ \mathcal{V}_1(a,b) + \int_0^\tau \left(u^{(3)}(s,a,b) + \mathbf{i}_1\right)^2 ds &\subseteq v^{(3)}(\tau,a,b) + \mathbf{j}_1 \end{aligned}$$

for all $a,\ b \in [-0.05, 0.05]$ and for all $\tau \in [0, 0.1]$.

Naive Taylor Model Method: Second Integration Step

Find $\mathbf{i}_1$ and $\mathbf{j}_1$ s.t.

$$\mathcal{U}_1(a,b) + \int_0^\tau \left(v^{(3)}(s,a,b) + \mathbf{j}_1\right)\, ds \quad \subseteq \quad u^{(3)}(\tau,a,b) + \mathbf{i}_1,$$

$$\mathcal{V}_1(a,b) + \int_0^\tau \left(u^{(3)}(s,a,b) + \mathbf{i}_1\right)^2 ds \quad \subseteq \quad v^{(3)}(\tau,a,b) + \mathbf{j}_1$$

for all a, $b \in [-0.05, 0.05]$ and for all $\tau \in [0, 0.1]$.

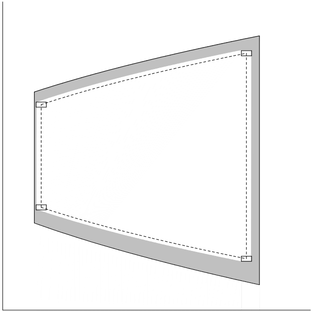
Since $\mathbf{i}_0$ and $\mathbf{j}_0$ are contained in $\mathcal{U}_1$ and $\mathcal{V}_1$, diameters of interval terms are nondecreasing!

Shrink Wrapping

Absorb interval term into polynomial part via shrink wrap factor q (Makino and Berz 2002):

$$\left.\begin{aligned}
\mathcal{U}(a,b) &:= 2 + 4a + \tfrac{1}{2}a^2 + [-0.2, 0.2],\\
\mathcal{V}(a,b) &:= 1 + 3b + 1ab + [-0.1, 0.1],\\
\mathcal{U}_{\mathrm{sw}}(a,b) &:= 2 + \frac{89}{20}a + \frac{89}{160}a^2,\\
\mathcal{V}_{\mathrm{sw}}(a,b) &:= 1 + \frac{287}{80}b + \frac{89}{80}ab.
\end{aligned}\right\}
\quad
\begin{aligned}
&a,\ b \in [-1,1],\\
&\left(q = \frac{89}{80}\right).
\end{aligned}$$

Shrink Wrapping



$$\begin{pmatrix} \mathcal{U} \\ \mathcal{V} \end{pmatrix} \textit{(white)} \quad \text{vs.} \quad \begin{pmatrix} \mathcal{U}_{\mathrm{sw}} \\ \mathcal{V}_{\mathrm{sw}} \end{pmatrix}.$$

Integration with Preconditioned Taylor Models

Preconditioned integration: flow at t_j:

$$\mathcal{U}_j = \mathcal{U}_{l,j} \circ \mathcal{U}_{r,j} = (p_{l,j} + \mathbf{i}_{l,j}) \circ (p_{r,j} + \mathbf{i}_{r,j}).$$

Purpose: stabilize integration similar to QR interval method.

Theorem (Makino and Berz 2004)

If the initial set of an IVP is given by a preconditioned Taylor model, then integrating the flow of the ODE only acts on the left Taylor model.

Integration with Preconditioned Taylor Models

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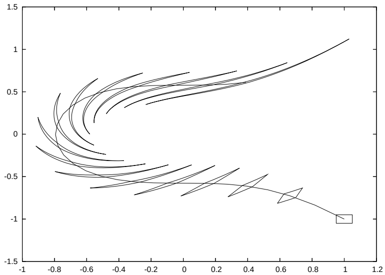
In practice:

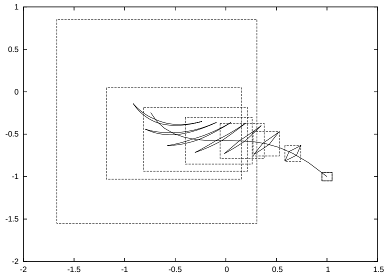
$$\mathcal{U}_j = (\widetilde{\mathcal{U}}_{l,j} \circ S_j) \circ (S_j^{-1} \circ \widetilde{\mathcal{U}}_{r,j}) \qquad S_j : \text{ scaling matrix.}$$

such that

$$\mathrm{Rg}\left(S_j^{-1} \circ \widetilde{\mathcal{U}}_{r,j}\right) \approx [-1, 1]^m.$$

Integration of Model Problem with COSY Infinity and AWA





Verified Integration of Linear ODEs

Interval Methods: Enclosure Representation

$$\begin{aligned} u(t_j;\mathbf{u}_0) &= \{u(t_j;u_0) \mid u_0 \in \mathbf{u}_0\} \\ &\subseteq \{u_j + S_j w + B_j r \mid w \in \mathbf{u}_0 - \mathrm{m}(\mathbf{u}_0),\ r \in \mathbf{r}_j\}, \end{aligned}$$

where

- $u_j, w, r \in \mathbb{R}^m,\ \mathbf{r}_j \in \mathbb{IR}^m$,
- $S_j, B_j \in \mathbb{R}^{m\times m}, \quad B_j$ nonsingular,
- $\{u_j + S_j w \mid w \in \mathbf{u}_0 - \mathrm{m}(\mathbf{u}_0)\}$: approximation to $u(t_j;\mathbf{u}_0)$,
- $\{B_j r \mid r \in \mathbf{r}_j\}$: bound on global error.

$j = 0$: $\quad u_0 = \mathrm{m}(\mathbf{u}_0), \quad \mathbf{r}_0 = 0, \quad S_0 = B_0 = I.$

Taylor Method for Linear Model Problem

Linear model problem:

$$u' = Au, \qquad (A \in \mathbb{R}^{m\times m},\ m \geq 2)$$

$$u(0) = u_0 \in \mathbf{u}_0.$$

Taylor method (constant order n, stepsize h):

$$u_j := Tu_{j-1} \quad , \quad j = 1, 2, \ldots.$$

$$\left(T = T_{n-1}(hA) = \sum_{\nu=0}^{n-1} \frac{(hA)^\nu}{\nu!} \right)$$

Taylor Method for Linear Model Problem

Linear model problem:

$$u' = Au, \qquad (A \in \mathbb{R}^{m\times m},\ m \geq 2)$$
$$u(0) = u_0 \in \mathbf{u}_0.$$

Interval Taylor method (constant order n, stepsize h):

$$u_j := Tu_{j-1} + z_j, \quad j = 1, 2, \ldots.$$

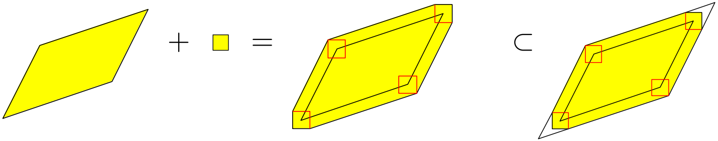
$$\left(T = T_{n-1}(hA) = \sum_{\nu=0}^{n-1} \frac{(hA)^\nu}{\nu!}; \quad z_j : \text{ local error} \right)$$

Propagation of the Global Error

$$\mathbf{r}_j = (B_j^{-1} T B_{j-1})\mathbf{r}_{j-1} + B_j^{-1}\big(\mathbf{z}_j - \mathrm{m}(\mathbf{z}_j)\big).$$

Required: B_j for tight enclosure:

$$\{\, TB_{j-1} r + z \mid r \in \mathbf{r}_{j-1},\ z \in \mathbf{z}_j - \mathrm{m}(\mathbf{z}_j)\} \subseteq \{B_j r \mid r \in \mathbf{r}_j\}.$$



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- Direct method: $B_j = I$.
- Parallelepiped method: $B_j = TB_{j-1}$.
- QR method: $B_j = Q_j,\ Q_j R_j = TB_{j-1}$.
- Blunting method: $B_j = TB_{j-1} + \varepsilon Q_j,\ \varepsilon > 0$.

Linear ODE: Naive TMM

Linear autonomous system ($A \in \mathbb{R}^{m\times m}$):

$$u' = A\,u, \quad u(0) \in \mathbf{u}_0 = \mathcal{U}_0.$$

Direct interval method ($\mathbf{z}_j$: local error, $T = \sum_{\nu=0}^{n} \frac{(hA)^\nu}{\nu!}$):

$$\mathbf{r}_j = T\mathbf{r}_{j-1} + \mathbf{z}_j - \mathrm{m}(\mathbf{z}_j), \quad j = 1, 2, \ldots.$$

Naive Taylor model method:

$$\mathcal{U}_j = T^j\,\mathcal{U}_0 + \sum_{k=1}^{j} (T\circ)^{j-k}\mathbf{i}_k, \quad j = 1, 2, \ldots,$$

where $(T\circ)^0\mathbf{x} := \mathbf{x}$, $(T\circ)^k\mathbf{x} := T \cdot \big((T\circ)^{k-1}\mathbf{x}\big)$, $k \in \mathbb{N}$.

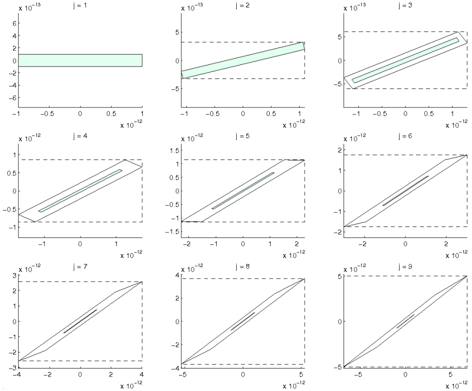
Wrapping Effect: Direct Method

$$\mathbf{r}_j = (B_j^{-1} T B_{j-1})\mathbf{r}_{j-1} + B_j^{-1}\big(\mathbf{z}_j - \mathrm{m}(\mathbf{z}_j)\big), \quad B_j = I:$$

$$\mathbf{r}_j = T\mathbf{r}_{j-1} + \mathbf{z}_j - \mathrm{m}(\mathbf{z}_j).$$

- Optimal coordinates for local error.
- Bad coordinates for global error (rotation).

Wrapping Effect: Direct Method



(Plots by Ned Nedialkov)

Huge overestimations in general.

Linear ODE: Naive TMM with Shrink Wrapping

Linear autonomous system ($A \in \mathbb{R}^{m \times m}$):

$$u' = A\,u, \quad u(0) \in \mathbf{u}_0 = \mathcal{U}_0.$$

Parallelepiped method ($\mathbf{z}_j$: local error, $\mathbf{r}_0 := \mathbf{u}_0 - m(\mathbf{u}_0)$):

$$\mathbf{r}_j := \mathbf{r}_{j-1} + (T^{1-j})(\mathbf{z}_j - (m(\mathbf{z}_j)), \quad j = 1, 2, \ldots .$$

Naive Taylor model method with shrink wrapping:

$$d_j := \left\| \mathrm{w}\left(T^{-j}\,\mathbf{i}_j \right) \right\|_\infty, \quad q_j := 1 + d_j/2, \quad \widetilde{p}_{sw,j} := \left(\prod_{k=1}^{j} q_j \right) \widetilde{p}_0(x).$$

Wrapping Effect: Parallelepiped Method

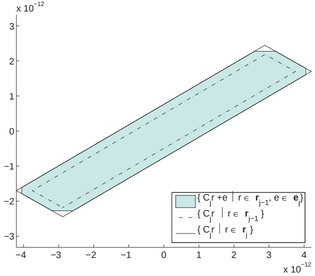
$$\mathbf{r}_j = (B_j^{-1} T B_{j-1})\mathbf{r}_{j-1} + B_j^{-1}\big(\mathbf{z}_j - \mathrm{m}(\mathbf{z}_j)\big), \quad B_j = TB_{j-1}:$$

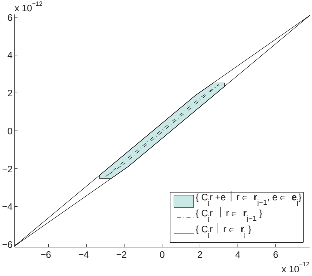
$$\mathbf{r}_j = \mathbf{r}_{j-1} + T^{-j}\big(\mathbf{z}_j - \mathrm{m}(\mathbf{z}_j)\big).$$

- Optimal coordinates for global error.
- Suitable coordinates for local error, if cond(T^j) is small.
- Bad coordinates for local error in presence of shear (T^j becomes singular for $j \to \infty$).

Wrapping Effect: Parallelepiped Method

$$B_j = TB_{j-1}$$





B_j often ill-conditioned, large overestimations.

Wrapping Effect: QR Method

Linear autonomous system ($A \in \mathbb{R}^{m \times m}$):

$$u' = A\,u, \quad u(0) \in \mathbf{u}_0 = \mathcal{U}_0.$$

QR method ($\mathbf{z}_j$: local error, $\mathbf{r}_0 := \mathbf{u}_0 - m(\mathbf{u}_0)$):

$$\mathbf{r}_j = R_j \mathbf{r}_{j-1} + Q_j^T \left(\mathbf{z}_j - \mathrm{m}(\mathbf{z}_j)\right), \quad j = 1, 2, \ldots .$$

- Suitable, but not optimal coordinates for global error.
- Kühn: Numerical example for exponential overestimation.
- Good coordinates for local error.
- Handles rotation, contraction, shear.

Preconditioned Taylor Model Method

Linear autonomous system ($A \in \mathbb{R}^{m \times m}$):

$$u' = A\,u, \quad u(0) \in \mathbf{u}_0 = \mathcal{U}_0.$$

Initial set: $p_{l,0}(x) = c_0 + C_0 x, \quad p_{r,0}(x) = x, \quad \mathbf{i}_{l,0} = \mathbf{i}_{r,0} = 0.$

jth initial set: $\mathcal{U}_j = (c_{l,j} + C_{l,j}\,x + \mathbf{i}_{l,j}) \circ (c_{r,j} + C_{r,j}\,x + \mathbf{i}_{r,j}),$

$$c_{l,j},\ c_{r,j} \in \mathbb{R}^m,\ \ C_{l,j},\ C_{r,j} \in \mathbb{R}^{m \times m}.$$

Integrated flow: $\widetilde{\mathcal{U}}_j := (Tc_{l,j} + TC_{l,j}\,x + \mathbf{i}_{l,j+1}) \circ (p_{r,j} + \mathbf{i}_{r,j}).$

Preconditioned Taylor Model Method

$C_{l,j+1}$ nonsingular: $\quad \widetilde{\mathcal{U}}_j = (Tc_{l,j} + C_{l,j+1}\,x + [0,0])$

$$\circ \left\{ C_{l,j+1}^{-1}\,TC_{l,j}\,c_{r,j} + C_{l,j+1}^{-1}\,TC_{l,j}\,C_{r,j}\,x + C_{l,j+1}^{-1}\,TC_{l,j}\,\mathbf{i}_{r,j} + C_{l,j+1}^{-1}\mathbf{i}_{l,j+1} \right\}$$

$$=: \quad (c_{l,j+1} + C_{l,j+1}\,x + [0,0]) \circ (c_{r,j+1} + C_{r,j+1}\,x + \mathbf{i}_{r,j+1}) =: \mathcal{U}_{j+1}$$

Global error:

$$\mathbf{i}_{r,j+1} := C_{l,j+1}^{-1}\,TC_{l,j}\,\mathbf{i}_{r,j} + C_{l,j+1}^{-1}\mathbf{i}_{l,j+1}, \quad j = 0, 1, \ldots .$$

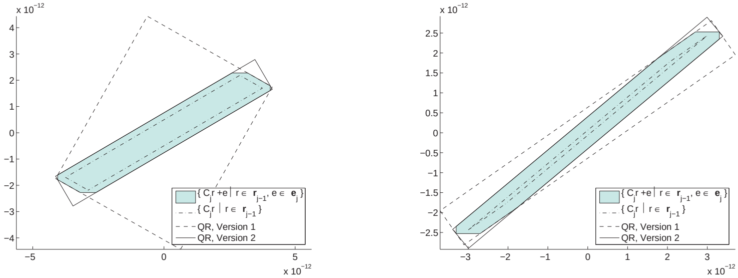
$C_{l,j+1} = TC_{l,j}$: parallelepiped preconditioning

$C_{l,j+1} = Q_j$: QR preconditioning

Other choices: curvilinear coordinates, blunting (Makino and Berz 2004)

Wrapping Effect: QR Method

$$B_j = Q_j, \quad Q_j R_j = T B_{j-1}$$



Overestimation depends on column permutations of B_{j-1}.

Example 1: Pure Contraction

$$A = \begin{pmatrix} -0.4375 & 0.0625 & -0.2652 \\ 0.0625 & -0.4375 & -0.2652 \\ -0.2652 & -0.2652 & -0.375 \end{pmatrix} \approx \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Method	Steps	$y_1(100)$	
AWA iv	216	1.4^{7301}_{5593}E-001	$\surd$
AWA pe	131	abort at $t = 52.6$	fail
AWA QR	216	1.4^{7301}_{5593}E-001	$\surd$
TM na	391	$[-2.4E+26, 2.4E+26]$	fail
TM sw	272	$[-2.3E+112, 2.3E+112]$	fail
TM QR	122	1.4^{7301}_{5593}E-001	$\surd$

$$(\mathbf{u}_0 = [0.999, 1.001] \cdot (1 \;\; 1 \;\; 1)^T)$$

Example 2: Pure Rotation

$$A = \begin{pmatrix} 0 & -0.7071 & -0.5 \\ 0.7071 & 0 & 0.5 \\ 0.5 & -0.5 & 0 \end{pmatrix} \approx \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Method	Steps	$y_1(100)$	
AWA iv	393	abort at $t = 76.5$	fail
AWA pe	449	$1.49^{522}_{222}\text{E}{+}000$	$\surd$
AWA QR	449	$1.49^{522}_{222}\text{E}{+}000$	$\surd$
TM na	396	$[-1.6\text{E}{+}45,\ 1.6\text{E}{+}45]$	fail
TM sw	343	$1.49^{522}_{222}\text{E}{+}000$	$\surd$
TM QR	343	$1.49^{522}_{222}\text{E}{+}000$	$\surd$

Example 3: Contraction and Rotation

$$A = \begin{pmatrix} -0.125 & -0.8321 & -0.3232 \\ 0.5821 & -0.125 & 0.6768 \\ 0.6768 & -0.3232 & -0.25 \end{pmatrix} \approx \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

Method	Steps	$y_1(100)$	
AWA iv	507	abort at $t = 85.5$	fail
AWA pe	404	abort at $t = 75.2$	fail
AWA QR	516	$1.34^{862}_{592}\text{E}+000$	$\surd$
TM na	397	$[-1.7\text{E}+55, 1.7\text{E}+55]$	fail
TM sw	357	$[-3.6\text{E}+106, 3.6\text{E}+106]$	fail
TM QR	362	$1.34^{862}_{592}\text{E}+000$	$\surd$

Summary

- Verified integration methods
- Interval methods vs. TM methods
- Performance for linear ODEs
- Future work: Analysis of TM methods for nonlinear ODEs

Thank you.

Questions or Remarks?