

A Note on a Sum Associated with the Generalized Hypergeometric Function

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Abstract

A new summation formula that is connected to the generalized hypergeometric function is presented. The formula can be applied in the Taylor series method for the solution of ordinary differential equations.

Key words: Explicit Summation, Generalized Hypergeometric Function

1 Introduction

This note is concerned with summation connected to the generalized hypergeometric function [1–4]. For some special cases, the summation formulas in this paper are easily proved with induction. Also, these special cases can be confirmed with the computer algebra system Maple [5]. For the general case, our summation formula appears to be new. In our experience, it can neither be confirmed with Maple 10 nor with Mathematica 5.1 [6].

While the summation formula has its own worth for practical calculations, we would like to draw the attention of the reader to the structure of our proof. It is based on mathematical induction with respect to two integer parameters i and k . This is, of course, a standard procedure. However, in the principal step of the proof, multiple occurrences of i are treated as independent parameters. One instance of i is replaced by a third, independent integer parameter μ . Mathematical induction is then performed with respect to μ (for fixed values of i and k). The desired result is finally obtained by setting $\mu = i - 1$. Although it seems strange that a more general claim is easier to verify, we have not been able to prove our assertion without introducing μ .

The paper is structured as follows. The new summation formula is stated in the next section. Its proof is carried out in four major steps in Section 3.

Straightforward proofs of special cases of the summation formula are given for completeness of the presentation, but are deferred to the appendix for better readability. In the final section, we report on some application of the summation formula.

2 Statement of Main Result

We recall Pochhammer's symbol [1, 6.1.22]

$$(k)_0 := 1, \quad (k)_i := k \cdot (k+1) \cdots (k+i-1) \quad \text{for } i, k \in \mathbb{N},$$

where $\mathbb{N} = \{1, 2, 3, \dots\}$ denotes the set of natural numbers. Pochhammer's symbol is connected to the faculty function via

$$(k)_i := \frac{(k+i-1)!}{(k-1)!} \quad \text{for } i, k \in \mathbb{N}.$$

Two special cases that appear frequently in this paper are

$$(k)_1 = k, \quad (1)_i = (2)_{i-1} = i! \quad \text{for } i, k \in \mathbb{N}. \quad (1)$$

The main result of this note is a summation formula for the function

$$M(i, k, m) := \sum_{j=1}^k \frac{(k-j+1)_{i-1}}{(j)_m},$$

where $i, k, m \in \mathbb{N}$ and $m > i$ are assumed.

$M(i, k, m)$ is connected to the generalized hypergeometric function ${}_pF_q(a, \rho, z)$ [3, Chap. 4] via

$$M(i, k, m) = \frac{\Gamma(k+i+1) {}_3F_2([1, 1, -k], [m+1, -k-i], 1)}{\Gamma(k+1) \Gamma(m+1)}.$$

This sum appears in the solution of certain ODEs with the Taylor series method, as pointed out in Section 4. In practical calculations, i and m are usually small (in practice, not larger than 10), but k often exceeds 100 and may sometimes even be larger than 1000. In this case, a summation formula involving substantially fewer than k summands is highly desired. We present a representation involving $i+1$ summands instead of k , which is useful if $i \ll k$ holds. To the author's knowledge, the existence of a closed expression for $M(i, k, m)$ with even fewer terms is an open question.

Theorem 1 For $i, k, m \in \mathbb{N}$ and $m > i$,

$$\begin{aligned} M(i, k, m) &= \sum_{j=1}^k \frac{(k-j+1)_{i-1}}{(j)_m} \\ &= \frac{(-1)^i (i-1)!}{(m-i)_i (k+i)_{m-i}} + \frac{(i-1)!}{(m-1)!} \sum_{\nu=0}^{i-1} \frac{(-1)^{i-1-\nu} (i-1-\nu)! (k)_\nu}{\nu! (m-i+\nu)_{i-\nu}}. \end{aligned} \quad (2)$$

The current versions of Maple and Mathematica are not able to confirm the latter identity in the generality expressed in Theorem 1. Only if one or several of the parameters i , k , or m are assigned fixed values, the identity is confirmed by these computer algebra systems. For example, Maple 10 confirms Theorem 1 for $M(2, k, m)$ or $M(k, 2, m)$, but not for $M(i, k, 2)$ nor for arbitrary $M(i, k, m)$. Maple recognize that $M(i, k, m)$ is connected to the generalized hypergeometric function, but it finds no suitable representation for the sum on the right hand side of (2), unless i or k are fixed.

With Mathematica 5.1, we have only succeeded in confirming Theorem 1 for fixed values of all parameters. The largest randomly chosen values tried were $(i, k, m) = (11, 2009, 113)$.

3 Proof of Theorem 1

The proof of Theorem 1 is given with mathematical induction with respect to i and k . i acts as the outer loop variable, k is used in the inner loop of the induction. m is considered an independent parameter. For $i \geq m$, the assertion is void. Hence, $m > i$ is assumed in all subsequent calculations.

To establish the basis for the induction, the special cases $i = 1$ and $k = 1$ are considered first.

3.1 Induction with respect to k for $i = 1$

For $i = 1$, Theorem 1 reads:

Proposition 1 For $k \in \mathbb{N}$, $m \geq 2$,

$$\sum_{j=1}^k \frac{1}{(j)_m} = \frac{1}{(m-1)(m-1)!} - \frac{1}{(m-1)(k+1)_{m-1}}.$$

Proof: For arbitrary $m \geq 2$, the proof of Proposition 1 is immediate with induction with respect to k . For completeness, it is given in the appendix. \square

3.2 Induction with respect to i for $k = 1$

For $k = 1$, Theorem 1 reads

$$\frac{(i-1)!}{m!} = \frac{(-1)^i(i-1)!}{(m-i)_i(i+1)_{m-i}} + \frac{(i-1)!}{(m-1)!} \sum_{\nu=0}^{i-1} \frac{(-1)^{i-1-\nu}(i-1-\nu)!}{(m-i+\nu)_{i-\nu}}. \quad (3)$$

Since

$$(i+1)_{m-i} = \frac{m!}{i!},$$

we have

$$\frac{(i-1)!}{m!} - \frac{(-1)^i(i-1)!}{(m-i)_i(i+1)_{m-i}} = \frac{(i-1)!}{m!} \left(1 - \frac{(-1)^i i!}{(m-i)_i} \right),$$

so that (3) is equivalent to

Proposition 2 For $m \geq 2$, $i = 1, 2, \dots, m-1$,

$$\sum_{\nu=0}^{i-1} \frac{(-1)^{i-1-\nu}(i-1-\nu)!}{(m-i+\nu)_{i-\nu}} = \frac{1}{m} \left(1 - \frac{(-1)^i i!}{(m-i)_i} \right).$$

Proof: For arbitrary $m \geq 2$, the proof of Proposition 2 is immediate with induction with respect to i . For completeness, it is given in the appendix. \square

3.3 A Formula for Partial Summation

The principal step in the proof of Theorem 1 is a summation formula for a sum that appears in the induction step in the next subsection. Because the derivation of this formula is long, we formulate it as a lemma.

Lemma 1 For $k \in \mathbb{N}$, $m \geq 2$, $i = 1, 2, \dots, m-1$, and $\mu = 1, 2, \dots, i-1$,

$$\begin{aligned} & \sum_{\nu=0}^{\mu} \frac{(-1)^{\nu}(i-1-\nu)!}{\nu!(m-i+\nu)_{i+1-\nu}} \left((i-\nu)(k)_{\nu} - m(k+1)_{\nu} \right) \\ &= \frac{(-1)^{\mu+1}(i-1-\mu)!(k+1)_{\mu}}{\mu!(m-i+\mu+1)_{i-\mu}}. \end{aligned}$$

Proof: The proof is with induction with respect to μ . For $\mu = 0$, we must show that

$$\frac{(i-1)!}{(m-i)_{i+1}}(i-m) = -\frac{(i-1)!}{(m-i+1)_i},$$

which is true since

$$(m-i)_{i+1} = (m-i) \cdot (m-i+1)_i.$$

Assuming that the assertion holds for a particular value of $\mu \leq i-2$, we obtain

$$\begin{aligned} & \sum_{\nu=0}^{\mu+1} \frac{(-1)^\nu (i-1-\nu)!}{\nu! (m-i+\nu)_{i+1-\nu}} \left((i-\nu)(k)_\nu - m(k+1)_\nu \right) \\ &= \frac{(-1)^{\mu+1} (i-1-\mu)! (k+1)_\mu}{\mu! (m-i+\mu+1)_{i-\mu}} \\ &+ \frac{(-1)^{\mu+1} (i-1-\mu-1)!}{(\mu+1)! (m-i+\mu+1)_{i-\mu}} \left((i-\mu-1)(k)_{\mu+1} - m(k+1)_{\mu+1} \right) \\ &= \frac{(-1)^{\mu+1} (i-2-\mu)!}{(\mu+1)! (m-i+\mu+1)_{i-\mu}} \\ &\quad \times \left((i-\mu-1)(\mu+1)(k+1)_\mu + (i-\mu-1)(k)_{\mu+1} - m(k+1)_{\mu+1} \right) \\ &= \frac{(-1)^{\mu+1} (i-2-\mu)! (k+1)_\mu}{(\mu+1)! (m-i+\mu+1)_{i-\mu}} \\ &\quad \times \left((i-1-\mu)(\mu+1) + (i-\mu-1)k - m(k+1+\mu) \right) \\ &= \frac{(-1)^{\mu+1} (i-2-\mu)! (k+1)_\mu}{(\mu+1)! (m-i+\mu+1)_{i-\mu}} (i-\mu-1-m)(\mu+1+k) \\ &= \frac{(-1)^{\mu+1} (i-2-\mu)! (k+1)_\mu (k+1+\mu)}{(\mu+1)! (m-i+\mu+1)(m-i+\mu+2)_{i-\mu-1}} (i-\mu-1-m) \\ &= \frac{(-1)^{\mu+2} (i-2-\mu)! (k+1)_{\mu+1}}{(\mu+1)! (m-i+\mu+2)_{i-\mu-1}}, \end{aligned}$$

which completes the proof of Lemma 1. \square

In the final step of the proof of Theorem 1, only the special case $\mu = i-1$ of Lemma 1 is required. However, we were not able to prove the special case other than introducing the parameter μ as above. For $\mu = i-1$, Lemma 1

reads

Corollary 1 For $k \in \mathbb{N}$, $m \geq 2$, and $i = 1, 2, \dots, m - 1$,

$$\sum_{\nu=0}^{i-1} \frac{(-1)^\nu (i-1-\nu)!}{\nu! (m-i+\nu)_{i+1-\nu}} \left((i-\nu)(k)_\nu - m(k+1)_\nu \right) = \frac{(-1)^i (k+1)_{i-1}}{(i-1)! m}.$$

3.4 Completion of the Proof of Theorem 1

From Proposition 1, we already know that the assertion holds for $i = 1$ and arbitrary k . Also, in Proposition 2, the assertion has been proved for $k = 1$ and arbitrary i . This provides the basis for the inner loop of the induction for arbitrary i . Hence, when passing from (i, k, m) to $(i, k+1, m)$ in the inner loop of the induction, we may assume that the assertion of Theorem 1 holds for all triples (ι, κ, μ) with $\mu \in \mathbb{N}$, $\mu > \iota$ and

$$\begin{cases} \kappa = 1, 2, \dots, k & \text{for } \iota = i, \\ \text{all } \kappa \in \mathbb{N} & \text{for } \iota < i. \end{cases} \quad (4)$$

It only remains to show (for $i \geq 2$ and $m > i$) that if Theorem 1 holds for some triple (i, k, m) , then it also holds for the triple $(i, k+1, m)$.

For $i \geq 2$, $m > i$ and $k \in \mathbb{N}$, we have

$$\begin{aligned} M(i, k+1, m) &= \sum_{j=1}^{k+1} \frac{(k+1-j+1)_{i-1}}{(j)_m} = \sum_{j=1}^{k+1} \frac{(k-j+2)_{i-2} (k-j+i)}{(j)_m} \\ &= \sum_{j=1}^{k+1} \frac{(k-j+1)(k-j+2)_{i-2}}{(j)_m} + (i-1) \sum_{j=1}^{k+1} \frac{(k-j+2)_{i-2}}{(j)_m} \\ &= \sum_{j=1}^k \frac{(k-j+1)(k-j+2)_{i-2}}{(j)_m} + (i-1) \sum_{j=1}^{k+1} \frac{(k-j+2)_{i-2}}{(j)_m} \\ &= \sum_{j=1}^k \frac{(k-j+1)_{i-1}}{(j)_m} + (i-1) \sum_{j=1}^{k+1} \frac{(k+1-j+1)_{i-2}}{(j)_m} \end{aligned}$$

By assumption (4), we may apply Theorem 1 on *both* sums. Hence,

$$\begin{aligned}
M(i, k+1, m) &= \sum_{j=1}^k \frac{(k-j+1)_{i-1}}{(j)_m} + (i-1) \sum_{j=1}^{k+1} \frac{(k-j+2)_{i-2}}{(j)_m} \\
&= \frac{(-1)^i (i-1)!}{(m-i)_i (k+i)_{m-i}} + \frac{(i-1)!}{(m-1)!} \sum_{\nu=0}^{i-1} \frac{(-1)^{i-1-\nu} (i-1-\nu)! (k)_\nu}{\nu! (m-i+\nu)_{i-\nu}} \\
&\quad + (i-1) \left\{ \frac{(-1)^{i-1} (i-2)!}{(m-i+1)_{i-1} (k+1+i-1)_{m-i+1}} \right. \\
&\quad \quad \left. + \frac{(i-2)!}{(m-1)!} \sum_{\nu=0}^{i-2} \frac{(-1)^{i-2-\nu} (i-2-\nu)! (k+1)_\nu}{\nu! (m-i+1+\nu)_{i-1-\nu}} \right\} \\
&= \frac{(-1)^i (i-1)!}{(m-i)_i (k+i)_{m-i}} + \frac{(-1)^{i-1} (i-1)!}{(m-i+1)_{i-1} (k+1+i-1)_{m-i+1}} \\
&\quad + \frac{(i-1)!}{(m-1)!} \left\{ \sum_{\nu=0}^{i-1} \frac{(-1)^{i-1-\nu} (i-1-\nu)! (k)_\nu}{\nu! (m-i+\nu)_{i-\nu}} \right. \\
&\quad \quad \left. + \sum_{\nu=0}^{i-2} \frac{(-1)^{i-2-\nu} (i-2-\nu)! (k+1)_\nu}{\nu! (m-i+1+\nu)_{i-1-\nu}} \right\} \\
&= \frac{(-1)^i (i-1)!}{(m-i+1)_{i-1} (k+i)_{m-i}} \left\{ \frac{1}{m-i} - \frac{1}{k+m} \right\} \\
&\quad + \frac{(i-1)!}{(m-1)!} \left\{ \frac{(k)_{i-1}}{(i-1)! (m-1)_1} \right. \\
&\quad \quad \left. + \sum_{\nu=0}^{i-2} \left(\frac{(-1)^{i-1-\nu} (i-1-\nu)! (k)_\nu}{\nu! (m-i+\nu)_{i-\nu}} + \frac{(-1)^{i-2-\nu} (i-2-\nu)! (k+1)_\nu}{\nu! (m-i+1+\nu)_{i-1-\nu}} \right) \right\} \\
&= \frac{(k)_{i-1}}{(m-1)(m-1)!} + \frac{(-1)^i (i-1)!}{(m-i+1)_{i-1} (k+i)_{m-i}} \cdot \frac{k+i}{(m-i)(k+m)} \\
&\quad + \frac{(-1)^{i-1} (i-1)!}{(m-1)!} \sum_{\nu=0}^{i-2} \left\{ \frac{(-1)^\nu (i-2-\nu)!}{\nu! (m-i+\nu)_{i-\nu}} \right. \\
&\quad \quad \left. \times \left((i-1-\nu)(k)_\nu - (m-i+\nu)(k+1)_\nu \right) \right\}
\end{aligned}$$

$$\begin{aligned}
&= \frac{(k)_{i-1}}{(m-1)(m-1)!} + \frac{(-1)^i(i-1)!}{(m-i+1)_{i-1}(m-i)} \cdot \frac{k+i}{(k+i)_{m-i}(k+m)} \\
&\quad + \frac{(-1)^{i-1}(i-1)!}{(m-1)!} \sum_{\nu=0}^{i-2} \left\{ \frac{(-1)^\nu(i-2-\nu)!}{\nu!(m-i+\nu)_{i-\nu}} \right. \\
&\quad \left. \times \left((i-1-\nu)(k)_\nu - (m-1)(k+1)_\nu + (i-\nu-1)(k+1)_\nu \right) \right\} \\
&= \frac{(k)_{i-1}}{(m-1)(m-1)!} + \frac{(-1)^i(i-1)!}{(m-i)_i(k+i+1)_{m-i}} \\
&\quad + \frac{(-1)^{i-1}(i-1)!}{(m-1)!} \sum_{\nu=0}^{i-2} \left\{ \frac{(-1)^\nu(i-2-\nu)!}{\nu!(m-i+\nu)_{i-\nu}} \right. \\
&\quad \left. \times \left((i-1-\nu)(k)_\nu - (m-1)(k+1)_\nu \right) \right\} \\
&\quad + \frac{(-1)^{i-1}(i-1)!}{(m-1)!} \sum_{\nu=0}^{i-2} \frac{(-1)^\nu(i-2-\nu)!(i-\nu-1)(k+1)_\nu}{\nu!(m-i+\nu)_{i-\nu}} \\
&= \frac{(k)_{i-1}}{(m-1)(m-1)!} + \frac{(-1)^i(i-1)!}{(m-i)_i(k+i+1)_{m-i}} \\
&\quad + \frac{(-1)^{i-1}(i-1)!}{(m-1)!} \sum_{\nu=0}^{i-2} \frac{(-1)^\nu(i-1-\nu)!(k+1)_\nu}{\nu!(m-i+\nu)_{i-\nu}} \\
&\quad + \frac{(-1)^{i-1}(i-1)!}{(m-1)!} \sum_{\nu=0}^{i-2} \left\{ \frac{(-1)^\nu(i-2-\nu)!}{\nu!(m-i+\nu)_{i-\nu}} \right. \\
&\quad \left. \times \left((i-1-\nu)(k)_\nu - (m-1)(k+1)_\nu \right) \right\}
\end{aligned}$$

Applying Corollary 1 (for $(i-1, m-1)$) on the latter sum, namely

$$\begin{aligned}
&\sum_{\nu=0}^{i-2} \left\{ \frac{(-1)^\nu(i-1-1-\nu)!}{\nu!(m-1-(i-1)+\nu)_{i-1+1-\nu}} \cdot \left((i-1-\nu)(k)_\nu - (m-1)(k+1)_\nu \right) \right\} \\
&= \frac{(-1)^{i-1}(k+1)_{i-2}}{(i-2)!(m-1)},
\end{aligned}$$

we obtain

$$\begin{aligned}
M(i, k+1, m) &= \frac{(k)_{i-1}}{(m-1)(m-1)!} + \frac{(-1)^i(i-1)!}{(m-i)_i(k+i+1)_{m-i}} \\
&+ \frac{(-1)^{i-1}(i-1)!}{(m-1)!} \sum_{\nu=0}^{i-2} \frac{(-1)^\nu(i-1-\nu)!(k+1)_\nu}{\nu!(m-i+\nu)_{i-\nu}} \\
&+ \frac{(-1)^{i-1}(i-1)!}{(m-1)!} \cdot \frac{(-1)^{i-1}(k+1)_{i-2}}{(i-2)!(m-1)} \\
&= \frac{(-1)^i(i-1)!}{(m-i)_i(k+i+1)_{m-i}} + \frac{(i-1)!}{(m-1)!} \sum_{\nu=0}^{i-2} \frac{(-1)^{i-1+\nu}(i-1-\nu)!(k+1)_\nu}{\nu!(m-i+\nu)_{i-\nu}} \\
&+ \frac{(k)_{i-1}}{(m-1)(m-1)!} + \frac{(i-1)(k+1)_{i-2}}{(m-1)(m-1)!}
\end{aligned}$$

Since

$$(k)_{i-1} + (i-1)(k+1)_{i-2} = (k+i-1)(k+1)_{i-2} = (k+1)_{i-1}$$

and

$$\frac{(k+1)_{i-1}}{(m-1)(m-1)!} = \frac{(i-1)!}{(m-1)!} \cdot \frac{(-1)^{i-1+(i-1)}(i-1-(i-1))!(k+1)_{i-1}}{(i-1)!(m-i+(i-1))_{i-(i-1)}},$$

the latter term can be absorbed in the above sum, so that finally

$$\begin{aligned}
M(i, k+1, m) &= \frac{(-1)^i(i-1)!}{(m-i)_i(k+i+1)_{m-i}} \\
&+ \frac{(i-1)!}{(m-1)!} \sum_{\nu=0}^{i-1} \frac{(-1)^{i-1+\nu}(i-1-\nu)!(k+1)_\nu}{\nu!(m-i+\nu)_{i-\nu}},
\end{aligned}$$

which completes the proof of Theorem 1.

4 Application

In [7], a method for the computation of error bounds for linear ODEs with polynomial coefficients was given. When the method is carried over to linear ODEs with analytic coefficients, sums of the form (2) appear, with $k \gg i$ (see [8] for the details). Theorem 1 can then be used to obtain error better estimates than were given in [8]. The method can be extended to linear and to some nonlinear systems of ODEs. Details are given in [9].

5 Conclusion

We have presented a new summation formula that is connected to the generalized hypergeometric function. The formula is useful when some parameter values are larger than others. It can be applied in the Taylor series method for the solution of ordinary differential equations.

6 Appendix

Proof of Proposition 1:

For arbitrary $m \geq 2$ and $k = 1$, Proposition 1 holds due to

$$\begin{aligned} \frac{1}{(1)_m} &\stackrel{(1)}{=} \frac{1}{m!} = \frac{1}{(m-1)(m-1)!} \cdot \frac{m-1}{m} \\ &= \frac{1}{(m-1)(m-1)!} - \frac{1}{(m-1)m!} \\ &\stackrel{(1)}{=} \frac{1}{(m-1)(m-1)!} - \frac{1}{(m-1)(2)_{m-1}} \end{aligned}$$

Assuming that Proposition 1 holds for some $k \in \mathbb{N}$, we obtain

$$\begin{aligned} \sum_{j=1}^{k+1} \frac{1}{(j)_m} &= \sum_{j=1}^k \frac{1}{(j)_m} + \frac{1}{(k+1)_m} \\ &= \frac{1}{(m-1)(m-1)!} - \frac{1}{(m-1)(k+1)_{m-1}} + \frac{1}{(k+1)_{m-1}(k+m)} \\ &= \frac{1}{(m-1)(m-1)!} - \frac{1}{(k+1)_{m-1}} \left(\frac{1}{m-1} - \frac{1}{k+m} \right) \\ &= \frac{1}{(m-1)(m-1)!} - \frac{1}{(k+1)_{m-1}} \cdot \frac{k+1}{(m-1)(k+m)} \\ &= \frac{1}{(m-1)(m-1)!} - \frac{1}{(m-1)(k+2)_{m-1}}, \end{aligned}$$

which completes the proof. □

Proof of Proposition 2:

For arbitrary $m \geq 2$ and $i = 1$, Proposition 2 reads

$$\frac{1}{(m-1)_1} = \frac{1}{m} \left(1 - \frac{(-1)^1}{(m-1)_1} \right),$$

that is

$$\frac{1}{m-1} = \frac{1}{m} \left(1 + \frac{1}{m-1} \right),$$

which is obviously true.

Assuming that Proposition 2 holds for some $i \in \mathbb{N}$, $i \leq m-2$, we obtain

$$\begin{aligned} & \sum_{\nu=0}^i \frac{(-1)^{i-\nu}(i-\nu)!}{(m-i-1+\nu)_{i+1-\nu}} \\ &= \frac{(-1)^i i!}{(m-i-1)_{i+1}} + \sum_{\nu=1}^i \frac{(-1)^{i-1-(\nu-1)}(i-1-(\nu-1))!}{(m-i+\nu-1)_{i-(\nu-1)}} \\ &= \frac{(-1)^i i!}{(m-i-1)_{i+1}} + \sum_{\nu=0}^{i-1} \frac{(-1)^{i-1-\nu}(i-1-\nu)!}{(m-i+\nu)_{i-\nu}} \\ &\stackrel{\text{Prop. 2}}{=} \frac{(-1)^i i!}{(m-i-1)_{i+1}} + \frac{1}{m} \left(1 - \frac{(-1)^i i!}{(m-i)_i} \right) \\ &= \frac{(-1)^i i! m}{(m-i-1)(m-i)_{i+1}} + \frac{1}{m} - \frac{(-1)^i i!}{(m-i)_{i+1}} \\ &= \frac{1}{m} + \frac{(-1)^i i!}{(m-i)_{i+1}} \left(\frac{m}{m-i-1} - 1 \right) \\ &= \frac{1}{m} + \frac{(-1)^i i!}{(m-i)_{i+1}} \cdot \frac{i+1}{m-i-1} \\ &= \frac{1}{m} - \frac{(-1)^{i+1}(i+1)!}{(m-i-1)_{i+2}} = \frac{1}{m} \left(1 - \frac{(-1)^{i+1}(i+1)!}{(m-i-1)_{i+1}} \right), \end{aligned}$$

which completes the proof. \square

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