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A complex mean value form for curves

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A new mean value form for analytic functions defined on curves in the complex plane is discussed.

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1. Introduction

The computation of a function value or of a range of a given function f on some interval X is a fundamental problem of numerical computations. However, since the floating point number set is finite, function values or ranges can rarely be computed exactly on a computer, due to the truncation and roundoff errors that appear in the calculation.

These errors can not be eliminated in general. Even their reduction, e.g., with multiple precision floating point arithmetic, is expensive. Nevertheless, there is a tool that enables the rigorous estimation of these errors, namely interval arithmetic [2,7,9] and its floating point implementation on computers. While analytic methods are employed to obtain rigorous bounds on truncation errors, roundoff errors in the calculation are handled by directed rounding operations [6]. Replacing the variable x in the definition of f by the domain X and evaluating this expression according to the rules of interval arithmetic, it is then possible to compute validated range bounds for a large class of functions.

The quality of the range bounds that are obtained by interval arithmetic evaluation of a function highly depends on the given function expression. Equivalence of expressions does not necessarily imply equivalent results, if these expressions are evaluated with interval arithmetic. For example, for $f(x) = x^2 - 2x$, the interval arithmetic evaluation of the expressions $F_1(X) := X * X - 2 * X$, $F_2(X) := X * (X - 2)$ and $F_3(X) := (X - 1)^2 - 1$ yields $F_1([0, 1]) = [-2, 1]$, $F_2([0, 1]) = [-2, 0]$, $F_3([0, 1]) = [-1, 0]$.

Moore [7] observed that centered forms gave particularly good range enclosures for intervals with small widths. While direct interval evaluation of f often converges linearly, quadratic convergence of the interval enclosures to the range can be reached with centered forms under mild assumptions on f [2,4,5,13].

For complex functions, Rokne [14,15] studied centered forms of polynomials and rational functions. These centered forms rely on manipulations of polynomials and are not applicable to other classes of analytic functions.

Petković and Petković [10,11] presented centered forms for the functions e^z, $\sin z$, $\cos z$, $\ln z$ and $z^{1/n}$, using circular complex arithmetic. However, the resulting range enclosures require the computation of explicit bounds for the Taylor remainder series of the respective function, which are not generally available.

For range bounds of real functions, Moore [7] also considered the mean value form. If $F'(X)$ is an interval that encloses the range of f' (the first derivative of f) on the interval X, then for each $c \in X$ the range of f on X is contained in

$$\mathrm{MVF}(X) \;:=\; f(c) + F'(X)(X - c).$$

The quadratic convergence of the mean value form for real functions has been treated in [2,3,12] (see also [13] and the references given there). Because the proof of the real mean value form relies on the real mean value theorem, it is not obvious that there is a complex mean value form. Nevertheless, the complex mean value form was developed in [8], for rectangular and for circular complex arithmetic. Conditions for quadratic convergence and inclusion isotonicity of the complex mean value form were also given in [8].

In this paper, we study the mean value form for analytic functions defined on curves. After a short introduction to interval arithmetic, a new mean value form for curves is developed in section 3.

2. Interval computations

Complex numbers are denoted by $z = x + \imath y$, where $x = \mathrm{Re}\, z$, $y = \mathrm{Im}\, z$. Analogously, $\zeta = \xi + \imath\eta$ is used. For a complex analytic function f we frequently write

$$f(z) = u(x, y) + \imath v(x, y),$$

where $u(x, y) = \mathrm{Re}\, f(z)$, $v(x, y) = \mathrm{Im}\, f(z)$. If f is analytic at $z_0 = x_0 + \imath y_0$ then [1, p. 24]

$$f'(z_0) \;=\; u_x(x_0, y_0) + \imath v_x(x_0, y_0). \tag{1}$$

Rectangular complex interval arithmetic. A real interval X is defined by a pair of two real numbers $\underline{x}, \overline{x}$ with $\underline{x} \leqslant \overline{x}$: $X = [\underline{x}, \overline{x}]$. A rectangular complex interval Z is defined by a pair of two real intervals X and Y:

$$Z = X + \imath Y, \quad Z = \{z = x + \imath y \mid x \in X,\ y \in Y\}.$$

The sets of real or complex intervals are denoted by $\mathbb{IR}$ or $\mathbb{IC}$, respectively.

Addition and multiplication of rectangular complex intervals are given by the following definition [2, definition 5.3]: Let $Z_1 = X_1 + \imath Y_1$ and $Z_2 = X_2 + \imath Y_2$ be two complex intervals, then

$$\begin{aligned} Z_1 \pm Z_2 &:= X_1 \pm X_2 + \imath(Y_1 \pm Y_2), \\ Z_1 \cdot Z_2 &:= X_1 Y_1 - X_2 Y_2 + \imath(X_1 Y_2 + X_2 Y_1). \end{aligned} \tag{2}$$

The division of complex intervals can be defined in several ways. In this paper, we use

$$\frac{Z_1}{Z_2} := \frac{X_1 Y_1 + X_2 Y_2}{Y_1 Y_1 + Y_2 Y_2} + \imath \frac{X_2 Y_1 - X_1 Y_2}{Y_1 Y_1 + Y_2 Y_2}, \tag{3}$$

provided that $0 \notin Y_1 Y_1 + Y_2 Y_2$.

For the *width* of a rectangular complex interval $Z = X + \imath Y$ we define

$$w(Z) := (\overline{x} - \underline{x}) + (\overline{y} - \underline{y}).$$

The *distance* of two rectangular complex intervals is measured by

$$q(Z_1, Z_2) := \max\{|\underline{x_2} - \underline{x_1}|, |\overline{x_2} - \overline{x_1}|\} + \max\{|\underline{y_2} - \underline{y_1}|, |\overline{y_2} - \overline{y_1}|\}.$$

Functions. For $D \subseteq \mathbb{C}$, the range of a function $f : D \to \mathbb{C}$ is denoted by $f(D)$, that is $f(D) := \{f(z) \mid z \in D\}$.

An *inclusion function* F of a given function f is an interval function (an expression that can be evaluated according to the rules of interval arithmetic) that encloses the range of f on all intervals $Z \subseteq D$:

$$F(Z) \supseteq f(Z) \quad \text{for all } Z \subseteq D.$$

The mean value form for rectangular complex intervals. In [8], the following theorem was proved:

Theorem 1 (Mean value form for rectangular complex intervals).

(a) Let $f(z) = u(x, y) + \imath v(x, y)$ be an analytic function in a domain $D \subseteq \mathbb{C}$, let $Z = X + \imath Y \subseteq D$ be a rectangular complex interval and let $\zeta = \xi + \imath\eta$ be a point in Z. Further, let $F'(Z)$ denote a rectangular complex interval that encloses the range of f' on Z. Then for the range $f(Z)$ the following inclusion holds:

$$f(Z) \subseteq \mathrm{MVF}(Z) := f(\zeta) + F'(Z)(Z - \zeta). \tag{4}$$

(b) The complex mean value form (4) is quadratically convergent, i.e. $q(f(Z), \mathrm{MVF}(Z)) = \mathrm{O}((w(Z))^2)$.

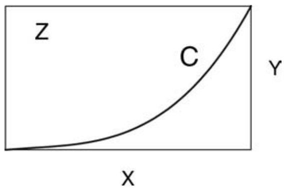

Figure 1. C with X, Y, Z.

3. A mean value form for curves

In this section, we address the following problem: Given a differentiable arc

$$C : z = \zeta(t) = \varphi(t) + \imath\psi(t), \quad t \in T = [\underline{t}, \overline{t}] \subset \mathbb{R},$$

and an analytic function

$$f(z) = u(x, y) + \imath v(x, y),$$

that is defined in a sufficiently large neighborhood of C, we want to enclose the range of

$$g(t) := f\big(\zeta(t)\big) =: r(t) + \imath s(t), \quad t \in T.$$

Let $X := \varphi(T)$, $Y := \psi(T)$, $Z := X + \imath Y$ (figure 1). If f is defined on all of Z then the mean value form (4) is applicable. However, (4) yields an enclosure of $f(Z)$, whereas we are only interested in an enclosure of $f(C)$.

Let $t_0 \in T$ be a fixed number. Furthermore, for $\tau_i \in T$ let $\zeta(\tau_i) = \xi_i + \imath\eta_i$, $i = 1, 2$. Then for $t \in T$, we apply the real MVT on r and s:

$$\begin{aligned}
g(t) - g(t_0) &= r(t) - r(t_0) + \imath\big(s(t) - s(t_0)\big) \\
&\overset{\text{MVT}}{=} r'(\tau_1)(t - t_0) + \imath s'(\tau_2)(t - t_0) \quad \text{for some } \tau_1, \tau_2 \in T \\
&= \Big(\text{Re}\,\big(f'\big(\zeta(\tau_1)\big)\zeta'(\tau_1)\big) + \imath\text{Im}\,\big(f'\big(\zeta(\tau_2)\big)\zeta'(\tau_2)\big)\Big)(t - t_0) \\
&\overset{(1)}{=} \Big(\text{Re}\,\Big(\big(u_x(\xi_1, \eta_1) + \imath v_x(\xi_1, \eta_1)\big)\big(\varphi'(\tau_1) + \imath\psi'(\tau_1)\big)\Big) \\
&\quad + \imath\text{Im}\,\Big(\big(u_x(\xi_2, \eta_2) + \imath v_x(\xi_2, \eta_2)\big)\big(\varphi'(\tau_2) + \imath\psi'(\tau_2)\big)\Big)\Big)(t - t_0) \\
&= \Big(u_x(\xi_1, \eta_1)\varphi'(\tau_1) - v_x(\xi_1, \eta_1)\psi'(\tau_1) \\
&\quad + \imath\big(u_x(\xi_2, \eta_2)\psi'(\tau_2) + v_x(\xi_2, \eta_2)\varphi'(\tau_2)\big)\Big)(t - t_0) \\
&= \big(u_x(\xi_1, \eta_1)\varphi'(\tau_1) + \imath v_x(\xi_2, \eta_2)\varphi'(\tau_2) \\
&\quad + u_x(\xi_2, \eta_2)\imath\psi'(\tau_2) + \imath v_x(\xi_1, \eta_1)\imath\psi'(\tau_1)\big)(t - t_0) \\
&\in \Big(\big(u_x(X, Y) + \imath v_x(X, Y)\big)\big(\varphi'(T) + \imath\psi'(T)\big)\Big)(T - t_0) \\
&\subseteq \big(F'(Z)\zeta'(T)\big)(T - t_0),
\end{aligned}$$

so that the following theorem holds:

Theorem 2. Let C: $z = \zeta(t)$, $t \in T \subset \mathbb{R}$ be a differentiable arc and let f be analytic in a neighborhood of C. Furthermore, let X, Y, Z, g be defined as above and let $F'(Z)$

denote a rectangular complex interval that contains $f'(Z)$. Then for arbitrary $t_0 \in T$ we have

$$g(T) \subseteq g(t_0) + \big(F'(Z)\zeta'(T)\big)(T - t_0). \tag{5}$$

(5) is called the mean value form for curves.

Remark. (5) is quadratically convergent if linearly convergent inclusion functions for f' and ζ' are employed.

The parentheses around $F'(Z)\zeta'(T)$ are essential for the quality of the enclosure. Complex multiplication is not associative in general [2, p. 55]. For our situation, we can show that multiplication in the given order is optimal:

Lemma 1. Let $W, Z \in \mathbb{IC}$, $T \in \mathbb{IR}$. Then

$$(W \cdot Z) \cdot T \;\subseteq\; W \cdot (Z \cdot T).$$

Proof. We write $W = U + \imath V$, $Z = X + \imath Y$. Using the well known law of subdistributivity [2, p. 3], i.e.

$$A(B + C) \subseteq AB + AC \quad \text{for } A, B, C \in \mathbb{IR}, \tag{6}$$

we have

$$\begin{aligned}
&\big((U + \imath V) \cdot (X + \imath Y)\big) \cdot T \\
&\quad = T(UX - VY) + \imath T(UY + VX) \overset{(6)}{\subseteq} TUX - TVY + \imath(TUY + TVX) \\
&\quad = U(TX) + \imath V(\imath TY) + U(\imath TY) + \imath V(TX) \;=\; (U + \imath V) \cdot \big((X + \imath Y) \cdot T\big). \quad \square
\end{aligned}$$

Performing the multiplication in the order

$$F'(Z)\big(\zeta'(T)(T - t_0)\big) \tag{7}$$

inflates the enclosure. (7) is even worse than the rectangular mean value form (4). For $z_0 := \zeta(t_0)$, we have

$$\begin{aligned}
\zeta'(T)(T - t_0) &= \varphi'(T)(T - t_0) + \imath\psi'(T)(T - t_0) \\
&\overset{MVF}{\supseteq} \varphi(T) - \varphi(t_0) + \imath\big(\psi(T) - \psi(t_0)\big) \\
&= X + \imath Y - z_0 \;=\; Z - z_0,
\end{aligned}$$

so that the rectangular mean value form (4) is contained in (7).

The two different range enclosures (4) and (5) can not be favored against each other. This is illustrated with the following example. Consider the function

$$f(z) = -\imath \ln z,$$

defined on the curve

$$C\colon\ \zeta(t) = \mathrm{e}^{\imath t}, \quad \frac{\pi}{6} \leqslant t \leqslant \frac{\pi}{3}.$$

We first apply the rectangular mean value form (4). Using

$$Z = \left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right](1+\imath), \qquad z_0 = \frac{\sqrt{2}}{2}(1+\imath)$$

we obtain

$$\begin{aligned}
f(C) &\subseteq f(z_0) + F'(Z)(Z - z_0) \\
&= f(z_0) - \frac{\imath}{Z}(Z - z_0) \\
&= \frac{\sqrt{2}}{2}(1+\imath) - \frac{\imath}{[1/2, \sqrt{3}/2] + \imath[1/2, \sqrt{3}/2]}(Z - z_0) \\
&= \frac{\sqrt{2}}{2}(1+\imath) + \frac{1/2, \sqrt{3}/2] - \imath[1/2, \sqrt{3}/2]}{[1/4, 3/4] + [1/4, 3/4]}(Z - z_0) \\
&= \frac{\sqrt{2}}{2}(1+\imath) + \left(\left[\frac{1}{3}, 3\right] + \imath\left[-\sqrt{3}, -\frac{1}{3}\right]\right)(Z - z_0) \\
&= \frac{\sqrt{2}}{2}(1+\imath) + \left(\left[\frac{1}{3}, 3\right] + \imath\left[-\sqrt{3}, -\frac{1}{3}\right]\right)\left(\left[\frac{1-\sqrt{2}}{2}, \frac{\sqrt{3}-\sqrt{2}}{2}\right]\right. \\
&\quad \left. + \imath\left[\frac{1-\sqrt{2}}{2}, \frac{\sqrt{3}-\sqrt{2}}{2}\right]\right) \\
&= \frac{\sqrt{2}}{2}(1+\imath) + \left[\sqrt{3}-\sqrt{6}, 3-\sqrt{6}\right] + \imath\left[\frac{-3+\sqrt{3}}{2}, \frac{3-\sqrt{3}}{2}\right] \\
&\approx \frac{\sqrt{2}}{2}(1+\imath) + [-0.551, 0.551] + \imath[-0.634, 0.634].
\end{aligned}$$

For the mean value form for curves (5), we have

$$\zeta'(T) = \imath \mathrm{e}^{\imath T} = \imath Z,$$

so that

$$\begin{aligned}
f(C) &\subseteq f(z_0) + \big(F'(Z)\imath Z\big)(T - t_0) \\
&= \frac{\sqrt{2}}{2}(1+\imath) + \left(\imath\left(\left[\frac{1}{3}, \sqrt{3}\right] + \imath\left[-\sqrt{3}, -\frac{1}{3}\right]\right) \cdot \left(\left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right] + \imath\left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]\right)\right) \\
&\quad \times \left[-\frac{\pi}{12}, \frac{\pi}{12}\right] \\
&= \frac{\sqrt{2}}{2}(1+\imath) + \left[-\frac{\pi}{9}, \frac{\pi}{9}\right] + \imath\left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \\
&\approx \frac{\sqrt{2}}{2}(1+\imath) + [-0.349, 0.349] + \imath[-0.785, 0.785].
\end{aligned}$$

While (4) gives a better enclosure of the imaginary part of the range, (5) yields a better enclosure of its real part. Both mean value forms are improved by intersecting (4) and (5). The additional costs of computing both mean value forms and their intersection (instead of computing only one of them) are usually low compared to the computation of $F'(Z)$.

4. Conclusion

A new mean value form for analytic functions on curves in the complex plane has been presented, which can improve the range enclosures derived from the ordinary mean value form. Both mean value forms can be computed simultaneously at few additional costs. The optimal range enclosure is then obtained by intersection.

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