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On Mean Value Forms for Complex Functions[1]

The mean value form is a well–known interval arithmetic means to compute tight range bounds for real functions. In this paper, the mean value form is extended for the computation of range bounds for complex analytic functions, both for rectangular and for circular complex intervals. Quadratic convergence and inclusion isotonicity of the complex mean value forms are discussed.

1. Ranges and inclusion functions

We assume that the reader is familiar with interval arithmetic as presented in [1]. Real bounded and closed intervals are denoted by $A = [a] = [\underline{a}, \overline{a}]$, $B = [b] = [\underline{b}, \overline{b}]$, etc. A rectangular complex interval Z is defined by a pair of two real intervals X and Y: $Z = X + \imath Y = \{z = x + \imath y \mid x \in X,\ y \in Y\}$. Real or complex numbers are identified with point intervals. The *width* (diameter) of a real interval is given by $w(X) = \overline{x} - \underline{x}$, the *distance* between two real intervals is measured by $q(X_1, X_2) \ := \ \max\{\, |\underline{x_2} - \underline{x_1}|\,,\, |\overline{x}_2 - \overline{x}_1|\,\}$. The width of a rectangular complex interval $Z = X + \imath Y$ is $w(Z) = \max\{w(X), w(Y)\}$, the distance between two rectangular complex intervals is measured by $q(Z_1, Z_2) \ := \ \max\{\, |\underline{x_2} - \underline{x_1}|\,,\, |\overline{x_2} - \overline{x_1}|\,,\, |\underline{y_2} - \underline{y_1}|\,,\, |\overline{y_2} - \overline{y_1}|\,\}$. Circular complex intervals are defined by radius r and midpoint ζ: $Z = <\zeta, r> = \{z \in \mathbb{C} \mid |z - \zeta| \le r\}$.

Interval arithmetic is often used for range computations. Let f be a real or a complex function. Then the range of f on some real or complex interval Z is denoted by $f(Z)$, i.e. $f(Z) := \{f(z) \mid z \in Z\}$. An *inclusion function* F of a given function f on $D \subseteq \mathbb{C}$ is an interval function that encloses the range of f on all intervals $Z \subseteq D$. An interval function F is called *inclusion isotone* in D if $F(Z_1) \subseteq F(Z_2)$ holds for all $Z_1 \subseteq Z_2 \subseteq D$.

2. The real mean value form

Let $m(X) \in X \subseteq \mathbb{R}$ and let F' be an inclusion function for f'. Then

$$f(m(X)) + F'(X)(X - m(X)) \supseteq f(X)$$

is called the real mean value form. It is quadratically convergent, i.e.

$$q(f(X), f(c) + F'(X)(X - c)) \ = \ O((w(X))^2),$$

if f' is satisfies a Lipschitz condition and if F' is a linearly convergent inclusion function for f'. The real mean value form is inclusion isotone if F' is inclusion isotone and if $m(X)$ is chosen as the midpoint of X.

3. Complex mean value forms

The real mean value form is now extended to complex analytic functions. The proofs of the following theorems have been given in [2].

Theorem 1 (Rectangular complex mean value form). *Let $f(z) = u(x, y) + \imath v(x, y)$ be analytic in a domain $D \subseteq \mathbb{C}$, let $Z \subseteq D$ be a rectangular complex interval, and let $z_0 = x_0 + \imath y_0$ be a point in Z. Furthermore, let $F'(Z)$ denote a rectangular complex interval that encloses the range of f' on Z. Then it holds that*

$$f(Z) \subseteq f(z_0) + F'(Z)(Z - z_0).$$

Theorem 2 (Quadratic convergence of the rectangular complex mean value form). *Let U_x and V_x be linearly convergent inclusion functions for the partial derivatives $u_x(x, y)$ and $v_x(x, y)$, respectively, and let*

$$F'(Z) := U_x(X, Y) + \imath V_x(X, Y).$$

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Then the mean value form converges quadratically to the range $f(Z)$.

T h e o r e m 3 (Circular complex mean value form). *Let f be analytic in a domain $D \subseteq \mathbb{C}$ and let $Z = <\zeta, r> \subseteq D$. Let $F'(Z)$ denote a circular complex interval with $f'(Z) \subseteq F'(Z)$. Then it holds that*

$$f(Z) \subseteq f(\zeta) + F'(Z)(Z - \zeta).$$

T h e o r e m 4 (Quadratic convergence of the circular complex mean value form). *Let f be analytic in a domain $D \subseteq \mathbb{C}$ and let F' be an inclusion function for f' such that*

$$w(F'(Z)) = O(w(Z)) \quad \textit{for alle} \quad Z \subseteq D.$$

For $Z = <\zeta, r>$ let $F_m(Z)$ denote the smallest disc that encloses $f(Z)$. Then the mean value form converges at least quadratically in the sense that

$$w(f(\zeta) + F'(Z)(Z - \zeta)) - w(F_m(Z)) = O((w(Z))^2).$$

For rectangular complex intervals, inclusion isotone inclusion functions U_x and V_x of u_x and v_x yield an inclusion isotone mean value form of $f(z) = u(x, y) + \imath v(x, y)$ via

$$f(z_0) + (U_x(X, Y) + \imath V_x(X, Y))(Z - z_0).$$

For circular complex arithmetic, we have:

T h e o r e m 5 (Inclusion isotonicity of the circular complex mean value form). *The circular complex mean value form is inclusion isotone if F' is inclusion isotone.*

4. Numerical example

We compare the direct interval evaluation $F(Z)$ of

$$f(z) = \cos(z) - \cos(2z) + \imath\cos(3z) \; - \; \frac{10557 + 11697\imath - (12862 + 4320\imath)z + (423 + 3027\imath)z^2}{11.44 + 5.88\imath + (-1.442 + 5.948\imath)z + (-0.7408 + 0.0058\imath)z^2}$$

with the rectangular complex mean value form. The quadratic convergence of the mean value form is clearly observed in the following table.

ε	$F(Z)$ $f'(z_0) + F'(Z)(Z - z_0)$
$1.0E-2$	$[-1.38E+3, 1.57E+3] + \imath[-1.25E+3, 1.47E+3]$ $[-1.11E+2, 1.58E+2] + \imath[-1.35E+2, 1.35E+2]$
$1.0E-4$	$[11.24, 35.72] + \imath[-12.11, 12.38]$ $[23.43, 23.48] + \imath[0.120, 0.159]$
$1.0E-6$	$[23.33, 23.58] + \imath[0.0171, 0.2620]$ $[23.4527, 23.4530] + \imath[0.139471, 0.139661]$

$z_0 = -1 + 3\imath$, $Z = z_0 + (1 + \imath)[-\varepsilon, \varepsilon]$.

5. References

1 Alefeld, G., and Herzberger, J.: Introduction to Interval Computations, Academic Press, New York, 1983.
2 Neher, M.: The Mean Value Form for Complex Analytic Functions, Computing **67** (2001), 255–268.

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