Neher, M.

Improved Bounds for Taylor Coefficients of Analytic Functions[1]

The practical computation of verified bounds for Taylor coefficients of analytic functions is considered. Using interval arithmetic, the bounds are constructed from Cauchy's estimate and from some of its modifications. By employing the mean value form for intermediate function evaluations, the accuracy of the bounds is improved by several powers of ten, compared to earlier results.

1. Cauchy's Estimate and Some Modifications

In the following, let $r > 0$, let B be the disc $\{z : |z| < r\}$, and let C be the circle $\{z : |z| = r\}$.

A well known bound for the Taylor coefficients of an analytic function is Cauchy's estimate: a function $f(z)$ that is analytic in B and bounded on C has a power series expansion $f(z) = \sum_{j=0}^{\infty} a_j z^j, \ |z| \leq r$, for which $|a_j| \leq \frac{M(r)}{r^j}, \ j \in \mathbb{N}_0$ holds, where $M(r) := \max_{|z|=r} |f(z)|$.

Cauchy's estimate is sometimes very pessimistic. Then the following modifications can be useful to obtain better bounds:

Theorem 1. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j, \ |z| \leq r, \ r > 0$, be analytic in B and bounded on C. Let $p_l(z)$ denote a polynomial of degree l. Applying Cauchy's estimate to $f - p_l$ yields*

$$|a_j| \leq \frac{N(r,l)}{r^j} \quad \text{for } j > l, \qquad \text{where } N(r,l) := \max_{|z|=r} |f(z) - p_l(z)|.$$

If p_l is a good approximation to f in the maximum norm then $N(r,l)$ can be much smaller than $M(r)$.

Theorem 2. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j, \ |z| \leq r, \ r > 0$, be analytic in B and let the m–th derivative of f be bounded on C. Further let $P(j,m) := (j+1)\cdots(j+m), \ P(j,0) := 1$ for $m \in \mathbb{N}, \ j \in \mathbb{N}_0$. Applying Cauchy's estimate to $f^{(m)}$ yields*

$$|a_j| \leq \frac{U(r,m)r^m}{P(j-m,m)r^j} \quad \text{for } j \geq m, \qquad \text{where } U(r,m) := \max_{|z|=r} |f^{(m)}(z)|. \tag{1}$$

(1) is a significant improvement of Cauchy's estimate, because $P(j-m,m) = O(j^m)$ if $j \to \infty$.

Remark 1. *The latter estimates can be combined if a polynomial approximation to $f^{(m)}$ is used in the computation of U.*

2. Practical Computation of the Bounds

Numerical computations are usually subject to errors, such as discretization or roundoff errors. To restore exactness in computation, interval arithmetic [1] has been developed and implemented in programming languaguages like C–XSC [4]. In the following, we will use rectangular complex interval arithmetic for the validated computation of bounds for Taylor coefficients.

For the practical calculation of $M(r)$, C is covered by complex intervals $Z_k, \ k = 1, \ldots, k_{\max}$. With an inclusion function F for f, for each Z_k an interval $|F(Z_k)| = [\underline{F}_k, \overline{F}_k]$ that encloses the range $|f(Z_k)|$ of $|f|$ on Z_k is computed. Proper inclusion functions for complex standard functions (like, e.g., the exponential function) were presented in [2].

The maximum number $\overline{F}_k$ is a validated upper bound for $M(r)$. Accurate upper bounds are usually obtained if the diameters of the interval Z_k are sufficiently small. For this purpose, adaptive refinement of the covering is performed according to the well–known branch and bound methods of global optimization [3].

[1]Appeared in: PAMM **1**, 450-451, 2002

The same ideas also apply to the computation of $U(r)$ or $N(r)$. For the latter, however, a cancellation occurs exactly in the favorable case, when $p_l \approx f$. The true range of $f - p_l$ on Z_k is then often strongly overestimated if it is directly enclosed into $F(Z_k) - P_l(Z_k)$. For the accurate computation of $N(r)$, the mean value form is better suited. It can be shown [5] that the range of $f - p_l$ on Z_k is contained in the interval arithmetic evaluation of $f(\zeta_k) - p_l(\zeta_k) + (Z_k - \zeta_k)(f'(Z_k) - p_l'(Z_k))$, where ζ_k is an arbitrary point in Z_k. The mean value form is quadratically convergent and may improve $N(r)$ by several powers of ten, compared to direct range enclosures.

R e m a r k 2. *The mean value form can also be used in the computation of $M(r)$ or $U(r)$. But here the range overestimation of direct inclusion functions is less important, and the mean value form does not affect the computed bounds very much.*

3. Numerical Examples

For the computation of the above bounds, a computer program has been written in C–XSC. The results in the following table were obtained on a PC with a Linux environment. In the examples on the computation of $N(r)$, Taylor polynomials p_l of order l, expanded with respect to the origin, were used.

Bounds for $f_1(z) = e^z$						Bounds for $f_2(z) = \frac{\cos z}{z^2 + 101}$					
r	l	m	$M/N/U$	a_{100}	a_{1000}	r	l	m	$M/N/U$	a_{100}	a_{1000}
1	—	—	2.8E+00	2.8E+00	2.8E+00	1	—	—	1.6E−02	1.6E−02	1.6E−02
1	11	—	1.3E−06	1.3E−06	1.3E−06	1	12	—	7.1E−09	7.1E−09	7.1E−09
1	—	50	2.8E+00	9.2E−94	9.8E−150	1	—	50	3.2E+18	1.1E−75	1.2E−131
10	—	—	2.7E+04	2.7E−96	2.7E−996	5	—	—	1.1E+00	1.4E−70	1.2E−699
10	29	—	7.4E−01	7.4E−101	7.4E−1001	5	28	—	1.3E−05	1.6E−75	1.4E−704
10	—	50	2.7E+04	8.6E−140	9.2E−1096	5	—	50	2.5E+31	8.9E−98	8.0E−783
20	—	—	5.9E+08	4.7E−122	5.5E−1293	10	—	—	1.3E+04	1.3E−96	1.3E−996
20	44	—	6.3E+04	5.0E−126	5.9E−1297	10	—	30	6.7E+78	8.6E−50	1.1E−981
20	—	50	5.9E+08	1.7E−150	2.2E−1377	10	—	50	5.5E+136	1.8E−07	1.9E−963

The table shows the computed bounds for the Taylor coefficients of two analytic functions. N and U are sometimes much smaller than M. On the other hand, a good polynomial approximation of f requires high order Taylor polynomials if r is large, and the computation of N can then be expensive.

f_2 has a singularity at $z = \sqrt{101}i$ that is near the circle with radius 10. Nevertheless, the computation of M and U was feasible even for $r = 10$, though $U(10, m)$ gets large for increasing m. Here, Cauchy's estimate is only improved for Taylor coefficients a_j with sufficiently large indexes j.

Acknowledgements

The author wishes to thank Ingo Eble for the programming of the numerical software.

4. References

1 Alefeld, G., Herzberger, J.: Introduction to interval computations, Academic Press, New York, 1983.
2 Braune, K., Krämer, W.: High–accuracy standard functions for real and complex intervals, *in* Computerarithmetic: Scientific computation and programming languages (Kaucher, E., Kulisch, U., Ullrich, Ch., Eds.), pp. 81–114, Teubner, Stuttgart, 1987.
3 Kearfott, R. B.: Rigorous global search: Continuous problems, Kluwer, Dordrecht, 1996.
4 Klatte, R., Kulisch, U., Lawo, Ch., Rauch, M., Wiethoff, A.: C–XSC: A C++ class library for extended scientific computing, Springer, Berlin, 1993.
5 Neher, M.: The mean value form for complex analytic functions, Preprint 01/13, Fakultät für Mathematik, Universität Karlsruhe.

Dr. Markus Neher, Institut für Angewandte Mathematik, Universität Karlsruhe, D–76128 Karlsruhe, Germany.