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Validated Bounds for the Zeros of Airy Functions[1]

This paper is concerned with the computation of validated bounds of zeros of Airy functions and their derivatives. The zeros are computed with Newton's method. Validated function values of Airy functions are calculated numerically with an enclosure method, using an interval arithmetic on the computer. With the latter, all roundoff errors in the computation are also included in the result. The algorithm has been implemented and tested.

1. Introduction

The Airy functions Ai(x) and Bi(x) are special solutions of Airy's differential equation

$$u'' + xu = 0.$$

They appear in several problems in physics, and they are used in the asymptotic theory of differential equations ([4, Chap. 11]; see [1] for properties of Airy functions).

In [6], Vrahatis et al. used the topological degree to determine the number N^r of zeros of Ai(x) (or Bi(x), Ai'(x), or Bi'(x)) within a given interval $I = [a, b]$:

$$N^r = -\frac{\xi}{\pi} \int_a^b \frac{\mathrm{Ai}(z)\mathrm{Ai}''(z) - \mathrm{Ai}'^2(z)}{\mathrm{Ai}^2(z) + \xi^2 \mathrm{Ai}'^2(z)}\, dz + \frac{1}{\pi}\left[\arctan\left(\frac{\xi \mathrm{Ai}'(b)}{\mathrm{Ai}(b)}\right) - \arctan\left(\frac{\xi \mathrm{Ai}'(a)}{\mathrm{Ai}(a)}\right)\right],$$

where ξ is an arbitrary positive constant.

However, regardless of what approximation of the integral is used, the calculation of the topological degree requires many function values of Ai, Ai', and Ai". In the following, we will show that the expensive computation of the topological degree is unneccessary, and that at most two function values of Ai alone are sufficient to determine N^r.

2. Bounds of zeros of Airy functions

The zeros of Ai and Bi are denoted by a_n and b_n, respectively. They are all negative and ordered decreasingly. In [5], Pittaluga and Sacripante used asymptotic expansions due to Miller [2] to construct precise error bounds for these zeros. For $n = 1$, the bounds in [5] are significant to 3 decimal digits, whereas for $n > 15$, bounds with a maximal relative precision in the standard IEEE number format are obtained. Hence, if $I = [a, b]$ is given, the number of zeros within I can be determined from the bounds from [5] and at most two function values of Ai or Bi (at a and/or b, if the endpoints of I lie within the bounds), respectively.

Closer bounds for the zeros a_1 through a_{14} than those in [5] can be computed with Newton's method. Using the bounds from [5] as starting values, in our numerical examples only a few Newton steps had to be performed to get maximal precision in the IEEE number format. The same applies to the computation of zeros of Bi(x) and of the zeros of the derivatives Ai' and Bi', which are denoted by a'_n and b'_n, respectively. For the zeros of the derivatives, no highly precise bounds are known, but rough bounds that can be used as starting values of Newton's method are readily available.

What remains is the problem of calculating function values. For small absolute values of x, an enclosure method that was developed in [3] for the validated solution of linear differential equations with polynomial coefficients was used to compute highly accurate bounds of function values of Ai(x), Bi(x), Ai'(x), and Bi'(x) on the computer.

The method works as follws: Using Airy's differential equation, a Taylor series expansion of the respective Airy function is computed (these series are known to converge for all $x \in \mathbb{R}$). Approximate function values are calculated with finitely many summands of the series. The respective remainder series is bounded by a geometric series. For small arguments, the series is rapidly convergent and the implementation of the method on a computer is immediate. For larger arguments, the terms of the series may cancel out each other. In this case, a multiple precision arithmetic must be used on the computer to achieve tight bounds.

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To compute function values of Ai' and Bi', the Taylor series of Ai and Bi were differentiated. Again, the remainder terms of these series were estimated by suitable geometric series.

3. Numerical results

The computer program for the location of zeros of the Airy functions Ai(x), Bi(x), and of their derivatives was written in PASCAL–XSC. Using the machine interval arithmetic of PASCAL–XSC, all roundoff errors of the computation were included in the results.

To demonstrate the high accuracy of the enclosure method for the function values, we first present a table with validated function values. They were all computed with double precision.

Function values of Ai(x), Ai'(x):

x	Ai(x)	Ai'(x)
12.82877675286575	[-2.34E-14 , -2.17E-14]	$3.0076323475407^{23}_{12}$E+00
12.82877675286576	[8.63E-15 , 1.04E-14]	$3.0076323475407^{23}_{11}$E+00
–20.53733290767756	[-2.52E-14 , -2.43E-14]	$3.3830024933729^{30}_{22}$E+00
–20.53733290767757	[1.08E-14 , 1.18E-14]	$3.3830024933729^{30}_{22}$E+00
–20.18863150946337	$-7.496868636184^{196}_{211}$E-01	[-4.74E-14 , -4.36E-14]
–20.18863150946338	$-7.496868636184^{195}_{211}$E-01	[1.14E-13 , 1.18E-13]

In the following table, enclosures of some zeros are shown. Almost all enclosures are accurate to 16 decimal digits. In [6], e.g. $a_{20} \approx -20.5373329075$ was obtained, which is only correct to 11 digits.

Zeros of Ai(x), Bi(x), Ai'(x), and Bi'(x):

n	a_n	b_n	a'_n	b'_n
1	$-2.338107410459976^{6}_{8}$	$-1.173713222709912^{7}_{9}$	$-1.01879297164747^{0}_{2}$	$-2.294439682614412^{2}_{5}$
2	$-4.087949444130097^{0}_{3}$	$-3.271093302836635^{2}_{4}$	$-3.248197582179983^{5}_{8}$	$-4.073155089071 8^{27}_{30}$
3	$-5.520559828095555^{0}_{3}$	$-4.830737841662001^{5}_{8}$	$-4.820099211178873^{3}_{6}$	$-5.512395729663^{597}_{602}$
10	$-12.828776752865 7^{5}_{6}$	$-12.38641713858 27^{3}_{5}$	$-12.38478837184 57^{4}_{5}$	$-12.82725830917 72^{1}_{2}$
15	$-16.905633997429 9^{4}_{5}$	$-16.52141955063 43^{7}_{9}$	$-16.5205038254 33^{79}_{80}$	$-16.90475941188 96^{4}_{6}$
20	$-20.537332907677 5^{6}_{7}$	$-20.18924478539 62^{0}_{1}$	$-20.18863150946 33^{7}_{8}$	$-20.53674024145 32^{7}_{9}$

4. References

1 Abramowitz, M.; Stegun, I. A.: Handbook of Mathematical Functions. Dover Publ. Inc., New York, 1972.

2 Miller, J. C. P.: The Airy Integral. Mathematical Tables, Vol. B, British Association of Science, Cambridge University Press, Cambridge, 1946.

3 Neher, M.: Ein Einschließungsverfahren für $y^{(n)} = \sum_{i=0}^{n-1} p_i(x) y^{(i)} + p(x)$. ZAMM **75** (1995), S547–548.

4 Olver, F. W. J.: Asymptotics and special functions. Academic Press, New York, 1974.

5 Pittaluga, G.; Sacripante, L.: Inequalities for the Zeros of the Airy Functions. SIAM J. Math. Anal. **22** (1991), 260–267.

6 Vrahatis, M. N.; Ragos, O.; Zafiropoulos, F.; Grapsa, T.N.: *Locating and Computing Zeros of Airy Functions.* ZAMM **76** (1996), 419–422.

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