NEHER, M.

Impedance Reconstruction in an Inverse Sturm–Liouville Problem with Finite Data[1]

The impedance case of the inverse Sturm–Liouville problem with finite data is considered. An enclosure algorithm is presented and numerical examples are given.

1. Reconstruction of an Impedance

Consider the impedance case of the Sturm-Liouville problem with Dirichlet boundary conditions:

$$\begin{aligned} (p(x)u')' + \lambda p(x)u &= 0 \\ u(0) = u(\pi) &= 0, \end{aligned} \tag{1}$$

where $p(x)$ is assumed to be continuously differentiable in $[0, \pi]$, symmetric about $\frac{\pi}{2}$ (that is, $p(x) = p(\pi - x)$ holds for all $x \in [0, \pi]$) and positive, with $p(0) = 1$ (see eg. [2], Chap. 8, for basic properties and more general formulations of the problem). In the above setting, the eigenvalues can be interpreted as functionals of the impedance $p(x)$, denoted by $\lambda_i(p)$, $i \in \mathbb{N}$, in increasing order: $0 < \lambda_1(p) < \lambda_2(p) < \dots$.

Opposite to the computation of eigenvalues and eigenfunctions of (1) for a given impedance $p(x)$, the inverse problem is concerned with the reconstruction of $p(x)$ in (1) from spectral data. Since, in applications, only a few of the lowest eigenvalues can be measured, we consider the following inverse problem: for given numbers ν_i, $i = 1, \dots, n$, we seek an impedance $p(x) = p(x; a) := 1 + \sum_{j=1}^{n} a_j\, p_j(x)$ with $a = (a_j) \in \mathbb{R}^n$ and differentiable, symmetric *basis functions* $p_j(x)$, $j = 1, 2, \dots, n$, so that

$$\lambda_i(p(x; a)) = \nu_i \quad \text{for} \quad i = 1, 2, \dots, n. \tag{2}$$

Let $f = (f_i(a)) := (\lambda_i(p(x; a)) - \nu_i)$, then by the definition of f, $p(x; a)$ fulfills (2) iff $f(a) = 0$. f is a differentiable function with partial derivatives

$$\frac{\partial f_i}{\partial a_j}(a) = \int_0^\pi \left(u_i'^2(x; a) - \lambda_i u_i^2(x; a)\right) p_j(x)\, dx, \quad i, j = 1, 2, \dots, n,$$

where $u_i(x, a)$ is the normalized i–th eigenfunction of (1) belonging to $p(x; a)$, with $\int_0^\pi p(x; a) u_i^2(x; a)\, dx = 1$.

For special choice of the basis functions $p_j(x)$, the Jacobian of f at $a = 0$ is nonsingular. In this case, there is a neighbourhood of $a = 0$ (which corresponds to $p(x) \equiv 1$) in which the inverse problem (2) has a locally unique solution and in which Newton's method is applicable.

Solving $f(a) = 0$ with Newton's method, the iteration

$$\left.\begin{aligned} a^{(k+1)} &:= a^{(k)} - \left(\frac{\partial f_i}{\partial a_j}(a^{(k)})\right)^{-1} f(a^{(k)}) \\ p^{(k)}(x) &:= p(x; a^{(k)}) = 1 + \sum_{j=1}^{n} a_j^{(k)}\, p_j(x) \end{aligned}\right\}, \quad k = 0, 1, \dots,$$

requires the computation of the lowest n eigenvalues λ_i and eigenfunctions u_i of $p^{(k)}(x)$ in each iteration step. They can be computed by solving initial value problems with the methods described in [3] for related Sturm–Liouville problems.

2. Enclosing solutions

Due to discretization and rounding errors, the mathematical basis of our enclosure method is impaired if the computation is performed approximately on a digital computer.

[1]Appeared in: ZAMM **78** (S3), S1025-S1026, 1998

To handle disretizations errors, rigorous error bounds instead of approximate values must be computed in each of the calculations. With interval computations (as described in [1]), by employing interval Newton's method instead of Newton's method, the existence of a solution to (2) can be verified. Let $[a] = ([a_j])$ denote an interval vector. Then with interval Newton's method, instead of a real function, an interval function $[p](x) = p(x; [a]) := 1 + \sum_{j=1}^{n} [a_j]\, p_j(x)$ that contains a validated solution $p(x; a)$ of the inverse impedance problem (2) is computed.

The major steps of the enclosure algorithm are the following: First, for an interval vector $[a]$ that is supposed to enclose a solution vector a of (2) and for $i = 1, \ldots, n$, an enclosing interval $[\lambda_i]$ of the i-th eigenvalues of all $p(x; a) \in p(x; [a])$ is computed. Then, for all $a \in [a]$ and all $\lambda \in [\lambda_i]$, the solutions of the initial value problems

$$\begin{aligned} (p(x;a)u')' + \lambda p(x;a)u &= 0 \\ u(0) = 0, \quad u'(0) &= 1 \end{aligned}$$

are enclosed in an interval function $[u_i](x)$. An enclosure of the Jacobian is then obtained from

$$\frac{\partial f_i}{\partial a_j}([a]) \in \frac{\displaystyle\int_0^\pi \left([u_i]'^2(x) - [\lambda_i][u_i]^2(x)\right)\, p_j(x)\, dx}{\int_0^\pi p(x; [a])[u]_i^2\, dx}, \quad i, j = 1, 2, \ldots, n.$$

The enclosure of a solution is proved if in the final step of the iteration, the interval Newton operator maps $[a]$ into itself.

The calculations can be performed on a computer if the roundoff errors are enclosed automatically by a machine interval arithmetic as is supplied by programming languages like PASCAL–XSC, FORTRAN–XSC or C–XSC.

3. Numerical results

We present two examples in which the leading coefficients of the cosine Fourier series of the unknown impedance $p(x)$ are reconstructed from eigenvalue data. Since $p(0) = 1$, the usual cosine series representation is replaced by the equivalent representation $p(x; a) = 1 + \sum_{j=1}^{n} a_j \sin^2(jx)$.

Example 1: Validated enclosure of $p(x) = 1 - 0.9 \sin^2(2x)$ (reconstruction from 5 eigenvalues):

$p(x) \in 1$

+ [–2.136694253250389E–014 , 2.445076580085002E–014] $\sin^2(1x)$
+ [–9.000000000000076E–001 , –8.999999999999948E–001] $\sin^2(2x)$
+ [–2.136634546421748E–014 , 2.386020921317758E–014] $\sin^2(3x)$
+ [–6.665187758538558E–015 , 6.322739182794063E–015] $\sin^2(4x)$
+ [–5.131264439907449E–015 , 5.899708315428504E–015] $\sin^2(5x)$

Example 2: Validated enclosure of $p(x) = 1 + 10 \sin^2(5x)$ (reconstruction from 5 eigenvalues):

$p(x) \in 1$

+ [–1.544533837329674E–012 , 1.592597233517635E–012] $\sin^2(1x)$
+ [–1.139809443979903E–012 , 1.155232755450755E–012] $\sin^2(2x)$
+ [–1.147224683291774E–012 , 1.133244064910754E–012] $\sin^2(3x)$
+ [–1.614480947367529E–012 , 1.573029457182213E–012] $\sin^2(4x)$
+ [9.999999999999134E+000 , 1.000000000000093E+001] $\sin^2(5x)$

4. References

1 Alefeld, G., Herzberger, J.: Introduction to Interval Computations. Academic Press, New York, 1983.
2 Gladwell, G. M. L.: Inverse Problems in Vibration. Martinus Nijhoff Publishers, Dordrecht, 1986.
3 Neher, M.: Ein Einschließungsverfahren für das inverse Dirichletproblem. Thesis, Karlsruhe University, 1993.

Address: Dr. Markus Neher, Institute of Applied Mathematics, Karlsruhe University, D-76128 Karlsruhe, Germany.