NEHER, M.

Enclosing Power Series Solutions of ODEs[1]

The numerical computation of power series solutions of a class of ordinary initial value problems is investigated. Estimation of the remainder term of the series by a geometric series yields continuous enclosures of the desired solutions. In contrast to other enclosure methods, no a priori–bounds of the solution are required.

Guaranteed enclosures are obtained on the computer when all rounding errors are taken into account in the calculations. The applicability of the method is demonstrated with numerical examples.

1. Enclosure Theorem

In [1] and [2], enclosure methods for some linear initial value problems were presented. In this paper, the nonlinear case is treated.

Let $y(x)$ denote the solution of the initial value problem

$$y' = f(p(x), y), \quad y(0) = \eta; \quad y = (y_1, \dots, y_n), \quad p = (p_{11}, \dots, p_{nJ}), \tag{1}$$

where f_i is a polynomial in y, with analytic coefficient functions $p_{ij}(x)$:

$$f_i(x) = \sum_{j=1}^{J} p_{ij}(x) \prod_{l=1}^{n} y_l^{\alpha_{ijl}}, \qquad p_{ij}(x) = \sum_{k=0}^{\infty} b_{ijk}\, x^k,$$

$$J \in \mathbb{N}, \quad \alpha_{ijl} \in \mathbb{N}_0, \quad b_{ijk} \in \mathbb{R}, \quad |b_{ijk}| \leq \frac{B_{ij}}{R^k} \text{ for some } R, B_{ij} > 0.$$

It is well known that the solution of (1) can be written as a power series

$$y_i(x) := \sum_{k=0}^{\infty} a_{ik}\, x^k, \quad |x| < r \text{ for some } r > 0. \tag{2}$$

The a_{ik} are determined from recursion formulae, after inserting (2) into (1).

Theorem 1. *For given η, R and B_{ij}, positive constants r and A_i can be computed to ensure*

$$|a_{ik}| \leq \frac{A_i}{r^k} \quad \textit{for all } i = 1, \dots, n \textit{ and all } k \in \mathbb{N}.$$

Consequently, for $k \in \mathbb{N}$, $x = \omega r$, $\omega \in (0,1)$, the following continuous enclosure holds:

$$|y_i(x) - \sum_{l=0}^{k-1} a_{il}\, x^l| \leq \sum_{l=k}^{\infty} |a_{il}\, r^l|\, \omega^l \leq \frac{A_i\, \omega^k}{1-\omega}.$$

2. Computer Implementation and Numerical Examples

The computation of the approximate solution $\tilde{y}_i(x) = \sum_{l=0}^{k-1} a_{il} x^l$ and of the error bound $\frac{A_i \omega^k}{1-\omega}$ is immediate. For guaranteed results on a digital computer, however, roundoff errors must be included in the error bound. This task can be achieved by using machine interval arithmetic. To obtain a tight enclosure of the solution, a multi–precision arithmetic may be helpful, as roundoff errors propagate in the recursive calculation of a_{il}.

[1]Appeared in: ZAMM **77**(S2), S635-S636, 1997

Example 1: $y' = 1 + y^2$, $y(0) = 0$. (Exact solution: $y(x) = \tan(x)$.)

Estimation: In each integration step, starting with $y(\xi) = \eta$, let $y(x-\xi) = u(x) = \sum_{k=0}^{\infty} a_k x^k$ and solve the initial value problem $u' = 1 + u^2, u'(0) = \eta$. Simple calculations show that if $r, A \geq 0$ satisfy

$$|a_0| = |\eta| \leq A, \quad a_1 r = (1+\eta^2) r \leq A \quad \text{and} \quad r \leq \frac{1}{A}$$

then $|a_k r^k| \leq A$ for all $k \geq 0$.

x	$\tan(x)$	$[y](x)$
1.0	1.557 407 724 654 902 2	$1.557\ 407\ 724\ 654\ 90^{4}_{0}$
1.570 796 326 794 890	1.510 7 E+14	$1.^{846}_{507}$E+14
1.570 796 326 794 895	6.175 7 E+14	$^{3.0E+15}_{-2.8E+13}$

Example 2: **Lorenz system**

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y, & x(0) &= 0; & \sigma &= 10 \\ \dot{y} &= rx - y - xz, & y(0) &= 1; & r &= 28 \\ \dot{z} &= -bz + xy, & z(0) &= 0; & b &= \frac{8}{3}\ . \end{aligned}$$

Estimation: Let $x(t) = \sum_{k=0}^{\infty} a_k t^k$, $y(t) = \sum_{k=0}^{\infty} b_k t^k$, $z(t) = \sum_{k=0}^{\infty} c_k t^k$. Then if A, B, C and $R \geq 0$ satisfy

$$|a_j R^j| \leq A, \quad |b_j R^j| \leq B, \quad |c_j R^j| \leq C \quad \text{for} \quad j = 0, \ldots, k \tag{3}$$

then the same inequalities hold for $(k+1)$ if

$$\frac{\sigma(A+B)}{(k+1)A} \leq \frac{1}{R}, \quad \frac{rA+B}{(k+1)B} + \frac{AC}{B} \leq \frac{1}{R}, \quad \frac{b}{(k+1)} + \frac{AB}{C} \leq \frac{1}{R}\ . \tag{4}$$

Using the initial values of each integration step, the inequalities (3) for $j = 0$ yield A, B and C. Inserting these constants into (4), R is determined.

t	$[y](t)$
0.25	$11.673\ 664\ 293\ 077\ ^{75}_{41}$
0.5	$-6.620\ 759\ 1^{07}_{11}$
1.0	-9.3^{72}_{85}

The numerical examples were computed in PASCAL-XSC on a HP-Vectra (486/66XM) personal computer. The poor results of example 2 are due to the wrapping effect. Improvement of the enclosures will be the subject of future research.

3. References

1 Neher, M.: Ein Einschließungsverfahren für $y^{(n)} = \sum_{i=0}^{n-1} p_i(x) y^{(i)} + p(x)$; ZAMM **75** (1995), S547–S548.
2 Neher, M.: An Enclosure Method for Linear ODEs with Analytical Coefficients; To appear in ZAMM.

Address: Dr. Markus Neher, Institut für Angewandte Mathematik, Universität Karlsruhe, D-76128 Karlsruhe