Neher, M.

An Enclosure Method for Linear ODEs with Analytical Coefficients

The solution of an initial value problem for an ode with analytical coefficients is represented as a power series. Estimation of the remainder term of the series by a geometric series yields an enclosure of the solution. For large step sizes, tight continuous enclosures of the solution are obtained, without the need of a priori*–bounds of the solution.*

Using sequential defect correction, the necessary computations can be performed efficiently on a computer. Guaranteed enclosures are obtained on the computer when all rounding errors are taken into account in the computation.

1. Enclosure Theorem

Let $y(x)$ denote the solution of the initial value problem

$$y^{(n)} = \sum_{i=0}^{n-2} p_i(x)\, y^{(i)} + p(x), \quad y^{(i)}(0) \;=\; y_{io}, \quad i = 0,1,\ldots,n-1,$$

where

$$p_i(x) = \sum_{j=0}^{\infty} b_{ij}\, x^j, \quad p(x) = \sum_{j=0}^{\infty} b_j\, x^j, \quad |b_{ij}| \;\le\; \frac{B}{r^j}, \quad |b_j| \;\le\; \frac{\widehat{B}}{R^j}, \quad R > r > 0, \quad B, \widehat{B} > 0.$$

The solution can be written as a power series $y(x) = \sum_{k=0}^{\infty} a_k\, x^k$, $|x| < r$, with

$$a_{k+n}\, x^{k+n} \;=\; \sum_{i=0}^{n-2} \sum_{j=0}^{k} \frac{x^{n-i}\, P(k-j,i)\, b_{ij}\, x^j}{P(k,n)}\; a_{k+i-j}\; x^{k+i-j} \;+\; \frac{b_k\, x^{k+n}}{P(k,n)},$$

$$P(k,i) := (k+1)\cdots(k+i), \quad P(k,0) := 1.$$

It can be shown that for $k \ge r$

$$|a_{k+n}\, r^{k+n}| \quad\le\; \max_{l=0}^{k+n-1} |a_l\, r^l|\; B \left(\frac{r^3\,(1+\ln{(n-2)})}{(k+n)\,(k+n-1)} \;+\; \frac{r^2}{(n-1)\,(k+n)} \right) \;+\; \frac{\widehat{B}\, r^{k+n}}{P(k,n) R^k}, \tag{1}$$

where the first term within the brackets is missing if $n = 2$.

T h e o r e m 1. *From (1), $k_0 \in \mathbb{N}$ can be computed so that*

$$|a_k\, r^k| \;\le\; A \;:=\; \max_{l=0}^{k_0+n-1} |a_l\, r^l| \quad \textit{for all} \;\; k \in \mathbb{N}_0.$$

Consequently, for all $m \ge k_0$,

$$|\, y(x) - \sum_{j=0}^{m-1} a_k\, x^k \,| \;\le\; \frac{A\,\omega^m}{1-\omega}, \quad \omega = \frac{x}{r}, \quad |x| < r.$$

2. Computer Implementation and Numerical Examples

The computation of an approximate solution $\tilde{y}(x) = \sum_{k=0}^{m-1} a_k x^k$ and of the error bound $\frac{A\omega^m}{1-\omega}$ is immediate. For guaranteed results on a digital computer, roundoff errors must be included in the error bound. This task can be achieved by using machine interval arithmetic. To obtain a reasonable enclosure of the solution, i.e. a tight error bound, however, it is crucial to use a multi–precision arithmetic, as roundoff errors propagate in the recursive computation of $a_k x^k$.

In our numerical examples, the *staggered correction*-format ([1]) was used to enclose $a_k x^k$ into a sum of L machine intervals: $a_k\, x^k \in \sum_{l=0}^{L} [d_{k,l}]$, where the intervals $[d_{k,l}]$ are computed with sequential defect correction:

$$[d_{k+n,0}] := \Big(\sum_{i=0}^{n-2} \sum_{j=0}^{k} P(k-j,i)\, b_{ij}\, x^{n-i+j} \sum_{l=0}^{L} [d_{k+i-j,l}] + b_k\, x^{k+n} \Big) \Big/ P(k,n)$$

For $\mu = 1, \ldots, L$:

$$\begin{aligned} [d_{k+n,\mu-1}] &:= \operatorname{mid}([d_{k+n,\mu-1}]) \\ [d_{k+n,\mu}] &:= \Big(\sum_{i=0}^{n-2} \sum_{j=0}^{k} P(k-j,i)\, b_{ij}\, x^{n-i+j} \sum_{l=0}^{L} [d_{k+i-j,l}] + b_k x^{k+n} - P(k,n) \sum_{\nu=0}^{\mu-1} [d_{k+n,\nu}] \Big) \Big/ P(k,n). \end{aligned}$$

By augmenting L during the computation, the precision of the enclosure can be accommodated to the requirements of the given problem. To gain accuracy, the sums and scalar products in the above formulae must be evaluated with a precise scalar product supplied by PASCAL-XSC or other languages suitable for verified computing.

Example 1: $y'' = y,\ y(0) = 1,\ y'(0) = -1.$

Exact Solution: $y(x) = e^{-x}$.

x	[y](x)	L	m	Time
100	3.720 075 976 020 83^{7}_{5}E-44	10	395	54
200	1.383 896 526 736 73^{8}_{7}E-87	21	748	292
300	5.148 200 222 412 01^{6}_{1}E-131	32	1106	735

Example 2: **Eigenvalues of**

$$\begin{aligned} -y'' + \cos(2x)\, y &= \lambda y \\ y(0) = y(\pi) &= 0. \end{aligned}$$

IVP:

$$\begin{aligned} &y'' = (\cos(2x) - \lambda)\, y \\ &y(0) = 0,\ y'(0) = 1. \end{aligned}$$

λ	$[y](\frac{\pi}{2})$	L	m	Time
16.008 310 459 709 47	−7.6...E−17 −6.9...E−16	2	210	6.32
16.008 310 459 709 48	2.6...E−16 2.6...E−17	2	214	6.60
100.001 262 636 893 5	1.3...E−15 1.5...E−16	2	270	10.05
100.001 262 636 893 6	−4.1...E−18 −1.4...E−16	2	279	10.60

Verified eigenvalue enclosures:

$$\begin{aligned} \lambda_4 &\in 16.008\,310\,459\,709\,4^{8}_{7}, \\ \lambda_{10} &\in 100.001\,262\,636\,893\,{}^{6}_{5}. \end{aligned}$$

The numerical examples were computed in PASCAL-XSC on a HP-Vectra (486/66XM) personal computer.

3. References

1 Stetter, J.: Sequential defect correction for high-accuracy floating-point arithmetic; in: Numerical Analysis (Proceedings, Dundee 1983), Lecture Notes in Mathematics **1066** (1984), 186–202.

Address: Dr. Markus Neher, Institut für Angewandte Mathematik, Universität Karlsruhe, D-76128 Karlsruhe